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A LYAPUNOV DESIGNED MODEL-

REFERENCED CONTROL SYSTEM WITH

SIGNUM FUNCTION ADAPTIVE CONTROL LAWS

BY

JAMES STEVEN EPSTEIN, 1943

A

THESIS

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ABSTRACT

An examination of a model-referenced adaptive control system designed to satisfy the requirements of Lyapunov's direct method is made. It is found that each adaptive control loop requires a multiplier for its implementation. A new design is proposed which replaces the multipliers in the control loops by switches, thereby gaining a significant hardware advantage. A first order system designed by the new method is simulated on an analog computer and some refinements are made. The method is then generalized to include n^{th} order systems. The poles of the model, however, are subject to some restrictions. Finally, the problems associated with extending the design to systems in which the model has arbitrary poles are discussed.

ACKNOWLEDGEMENTS

This thesis was carried out under the direction of Dr. John S. Pazdera, whose comments were very helpful in keeping the author on the right track.

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LIST OF SYMBOLS

r	command input
K_m	model gain parameter
K_p	plant gain parameter
Θ_m	model output
Θ_s	plant output
b_{mi}	model parameter
b_i	plant parameter
k_i	adjustable controller parameter
K_c	adjustable controller parameter
e	error, $\Theta_m - \Theta_s$
s	Laplace operator
β_i	derivative feedback loop gain
x_i	difference between model and controlled plant parameter
$\underline{x}$	column vector of x_i's
$\underline{f}_i$	column vectors of matrix $\underline{F}^T$
$\underline{F}$	matrix containing plant output, its derivatives and the command input in the last row
μ_i	adaptive loop gain
$\underline{M}$	diagonal matrix of μ_i's
V	Lyapunov function
λ_i	eigenvalues of $\underline{A}_m$
$\underline{\Lambda}$	diagonal matrix of λ_i's
$\underline{A}_m$	coefficient matrix of the model state vector

$\underline{B}_m$ coefficient matrix of the model forcing function

$\underline{A}_s$ coefficient matrix of the controlled plant state vector

$\underline{B}_s$ coefficient matrix of the controlled plant forcing function

$\underline{P}$ symmetric positive definite matrix

$\underline{N}$ symmetric positive definite matrix

τ dummy variable of integration

$\underline{q}_i$ column vectors of matrix $\underline{Q}$

$\underline{Q}$ coefficient matrix of $\underline{e}$ inside the saturation function

$\underline{C}$ diagonal matrix of negative real constants

$\underline{T}$ Vandermonde matrix

w_i a definite integral of a saturation function

$\underline{w}$ column vector of w_i's

$\underline{J}$ a matrix consisting of the real part of the eigenvalue on the diagonal and the imaginary part on the off diagonals

$\underline{S}$ matrix to transform $\underline{\Lambda}$ to $\underline{J}$

α saturation function gain parameter

I. INTRODUCTION

A. A History of the Use of Lyapunov's Direct Method as a Design Technique

One of the major problems the control system engineer encounters is that of stability. Many classical methods are available to aid in designing a stable system; however, until recently it was thought that complicated non-linear systems could not be handled by any exact methods. With the need for large non-linear control systems, coupled with more exacting specifications, came the realization that another analysis tool was required. The direct method of Lyapunov partially fulfilled this need, with one major drawback; it only provides sufficient conditions for stability in most cases. Failure to find a Lyapunov function does not indicate that the system is unstable, only that the engineer may not have chosen the proper Lyapunov function. For this reason, it has been said that the direct method of Lyapunov has more merit as a synthesis technique[1]. A system can be designed so that it satisfies the conditions of Lyapunov's direct method and its stability is thereby guaranteed. Hence, the often difficult, if not impossible, task of determining stability after design is eliminated.

The major impetus in the United States for design using the Lyapunov direct method came from a paper by Kalman and Bertram[2] published in 1960. In July, 1961, a dissertation written by Grayson[1] indicated the advantages and value of the direct method as a synthesis tool and developed a framework for many different techniques. One of these was further developed by Monopoli[3], who investigated its engineering aspects in detail. An excellent summary of the techniques developed up to 1965 can be found in "The Status of Synthesis Using Lyapunov's Method" by Grayson[4].

Further work in developing the direct method as a design technique was done by Shackcloth and Butchart[5], who were working under Parks in a study of the use of Lyapunov functions. Their first objective was to obtain stability bounds on an existing system using the direct method as an analysis tool. Having little success along these lines, they were diverted to developing a synthesis technique. Since their first paper was published in 1965, several extensions have been made by other authors as well as Shackcloth. One particularly noteworthy paper was written by Parks[6] in which he applies the synthesis technique based on Lyapunov's direct method to redesign systems developed by several other authors. A further extension on the work of Parks just mentioned was made by Phillipson[7], who proposed a modification to reduce system oscillations.

Although the Lyapunov direct method is a very powerful design technique, it too, as expected, has disadvantages. In many cases it is impossible to determine whether or not you have the best design. Also, since developing new design techniques using Lyapunov's direct method is relatively new, there is not as much past experience to draw on as with some other methods.

B. Background for Proposed Design

In the field of adaptive control, there are wide applications for the direct method of Lyapunov as a design technique. More specifically, this thesis will deal with a model-referenced adaptive control system in which the model is used as a reference to adjust the controller parameters to compensate for time varying or unknown plant parameters. This type of system has an advantage since explicit identification of the plant dynamics is unnecessary. However, the stability of a model-referenced system is often impossible to determine using classical techniques. Some work along this line has been done by Bongiorno[8]. On the other hand, if the system is designed to satisfy the conditions of Lyapunov's direct method, its stability is guaranteed.

The Lyapunov approach is taken by Shackcloth[9] to determine the adaptive control laws for a model-referenced system. An examination of these control laws indicates that each control loop requires an integrator and a multiplier. In order to reduce the cost of the adaptive system, this thesis will propose an alternate design in which the multiplier in each adaptive loop can be replaced by a switch.

Hopefully, this is only the first step toward the ultimate goal of completely digitizing the adaptive part of the system.

The new design will be presented for a first order system and then extended to a system of any order. There are some restrictions on the model which will be pointed out, along with the problems encountered trying to eliminate them. Finally, suggestions are made for further work.

II. EXAMINATION OF AN EXISTING MODEL-REFERENCED ADAPTIVE CONTROL SYSTEM DESIGN

A. Technique for Designing Adaptive Loops

The technique developed by Shackcloth[9] is based on the system in Figure 1. All controller parameters are adjustable by changing the respective value of k_1 through k_n as seen by examining the overall controlled plant transfer function, for constant plant and controller parameters, in Equation (1).

$$\frac{\Theta_s(s)}{r(s)} = \frac{K_c K_p}{s^n + (b_n + K_p k_n)s^{n-1} + \cdots + (b_1 + K_p k_1)} \tag{1}$$

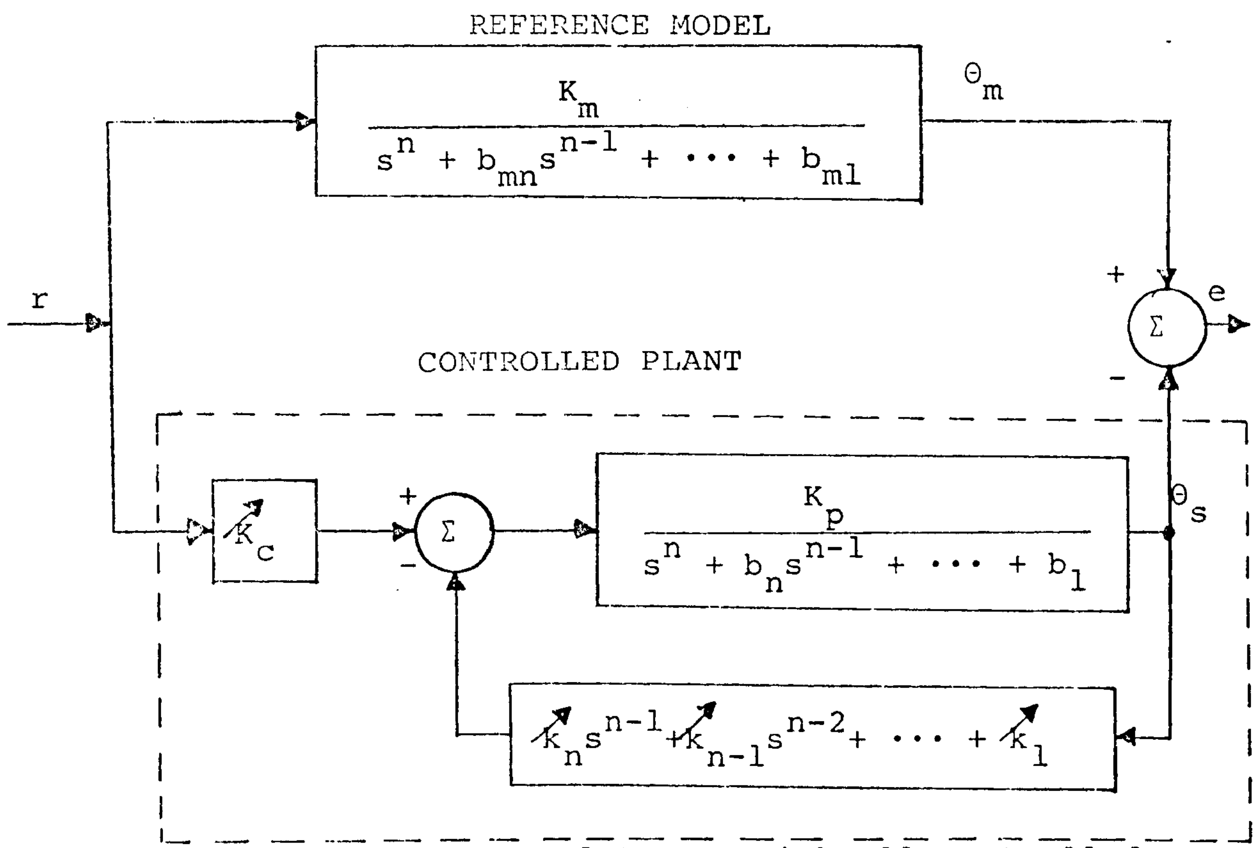

Figure 1. A Model Referenced System With All Controlled Plant Parameters Adjustable

The problem is to adjust the controlled plant parameters to be the same as the reference model parameters in such a manner that the overall system is stable. The approach taken by Shackcloth is to force a function, V, to be a Lyapunov function. V is chosen to be a function of the error between the output of the model and the output of the plant, and the difference between the model and controlled plant parameters. V is made a Lyapunov function by picking it to be positive definite and then making its derivative negative definite, or negative semi-definite, by properly choosing the adaptive control laws. The equations for the variable parameters k_1 through k_n and K_c will be referred to as the adaptive control laws. The system will then be asymptotically stable and the error will go to zero.

In deriving the adaptive control laws, the assumption made is that the plant parameters are constant during adaption. Hence, the results presented pertain directly to systems with step changes in parameters since the time that the parameters are changing is small. Another application would be to systems in which all parameters are constant, but cannot be measured.

To illustrate the derivation of the adaptive control laws more clearly, the system equations will be written in matrix form and some new matrices will be defined. In matrix notation, the equations for the model and controlled plant of Figure 1 can be written as follows:

$$\dot{\underline{\Theta}}_m = \underline{A}_m \underline{\Theta}_m + \underline{B}_m r \tag{2}$$

$$\dot{\underline{\Theta}}_s = \underline{A}_s \, \underline{\Theta}_s + \underline{B}_s \, r \tag{3}$$

The error is defined as the difference between the output of the model and the output of the plant, $\underline{e} \triangleq \underline{\Theta}_m - \underline{\Theta}_s$, then,

$$\dot{\underline{e}} = \dot{\underline{\Theta}}_m - \dot{\underline{\Theta}}_s = \underline{A}_m \underline{\Theta}_m - \underline{A}_s \underline{\Theta}_s + (\underline{B}_m - \underline{B}_s) r. \tag{4}$$

If $\underline{A}_m \, \underline{\Theta}_s$ is added to and subtracted from the right hand side of Equation (4),

$$\dot{\underline{e}} = \underline{A}_m \, \underline{e} + (\underline{A}_m - \underline{A}_s) \underline{\Theta}_s + (\underline{B}_m - \underline{B}_s) r; \tag{5}$$

where,

$$\underline{e} = (e \; \dot{e} \; \cdots \; e^{(n-1)})^T,$$

$$\underline{\Theta}_s = (\Theta_s \; \dot{\Theta}_s \; \cdots \; \Theta_s^{(n-1)})^T, \tag{6}$$

and,

$$\underline{\Theta}_m = (\Theta_m \; \dot{\Theta}_m \; \cdots \; \Theta_m^{(n-1)})^T.$$

In the companion form,

$$\underline{A}_m = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_{m1} & -b_{m2} & & \cdots & -b_{mn} \end{bmatrix}, \underline{B}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ K_m \end{bmatrix} \quad , \tag{7}$$

$$
\underline{A}_s = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 1 \\ -(b_1+K_pk_1) & -(b_2+K_pk_2) & \cdots & & -(b_n+K_pk_n) \end{bmatrix}, \underline{B}_s = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ K_cK_p \end{bmatrix}, \tag{8}
$$

and,

$$
\dot{\underline{e}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & \cdots & & 1 \\ -b_{m1} & -b_{m2} & & \cdots & & -b_{mn} \end{bmatrix} \underline{e}
$$

$$
+ \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & \cdots & 0 \\ (b_1+K_pk_1-b_{m1}) & (b_2+K_pk_2-b_{m2}) & \cdots & & & (b_n+K_pk_n-b_{mn}) \end{bmatrix} \underline{\Theta}_s
$$

$$
+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ K_m-K_cK_p \end{bmatrix} r. \tag{9}
$$

Let the vector $\underline{x}$ represent the difference between the reference model and controlled plant parameters,

$$\underline{x} = [x_1 \; x_2 \; \cdots \; x_{n+1}]^T$$

and

$$\begin{aligned} x_1 &= b_1 + K_p k_1 - b_{m1} \\ x_2 &= b_2 + K_p k_2 - b_{m2} \\ &\;\;\vdots \\ x_n &= b_n + K_p k_n - b_{mn} \\ x_{n+1} &= K_m - K_c K_p . \end{aligned} \tag{10}$$

Next, an n by n + 1 matrix $\underline{F}$ is defined such that:

$$\underline{F} = [\underline{f}_1 \; \underline{f}_2 \; \cdots \; \underline{f}_n]^T;$$

where,

$$\underline{f}_1 = \underline{f}_2 = \cdots = \underline{f}_{n-1} = \underline{0} \tag{11}$$

and,

$$\underline{f}_n = [\Theta_{s1} \; \Theta_{s2} \; \cdots \; \Theta_{sn} \; r]^T .$$

The error equation can now be written as:

$$\underline{\dot{e}} = \underline{A}_m \, \underline{e} + \underline{F} \, \underline{x} . \tag{12}$$

The V function chosen by Shackcloth[9] is:

$$V = \underline{e}^T \, \underline{P} \, \underline{e} + \frac{x_1^2}{\mu_1} + \frac{x_2^2}{\mu_2} + \cdots + \frac{x_{n+1}^2}{\mu_{n+1}} , \tag{13}$$

where $\underline{P}$ is a positive definite matrix to be specified later, and the μ_i's are positive adaptive loop gains.

Let $\underline{M}$ be a diagonal matrix,

$$\underline{M} = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \\ \cdot & & & \cdot \\ 0 & \cdots & & \mu_{n+1} \end{bmatrix} . \tag{14}$$

Now,

$$V = \underline{e}^T \underline{P}\ \underline{e} + \underline{x}^T \underline{M}^{-1} \underline{x} \tag{15}$$

and,

$$\dot{V} = \dot{\underline{e}}^T \underline{P}\ \underline{e} + \underline{e}^T \underline{P}\ \dot{\underline{e}} + \dot{\underline{x}}^T \underline{M}^{-1} \underline{x} + \underline{x}^T \underline{M}^{-1} \dot{\underline{x}}. \tag{16}$$

The substitution of $\dot{\underline{e}}$ from Equation (12) yields,

$$\dot{V} = \underline{e}^T \underline{A}_m^T \underline{P}\ \underline{e} + \underline{x}^T \underline{F}^T \underline{P}\ \underline{e} + \underline{e}^T \underline{P}\ \underline{A}_m \underline{e} + \underline{e}^T \underline{P}\ \underline{F}\ \underline{x}$$
$$+ \dot{\underline{x}}^T \underline{M}^{-1} \underline{x} + \underline{x}^T \underline{M}^{-1} \dot{\underline{x}} \ . \tag{17}$$

When the terms of Equation (17) are combined:

$$\dot{V} = \underline{e}^T [\underline{A}_m^T \underline{P} + \underline{P}\ \underline{A}_m]\underline{e} + 2\underline{x}^T [\underline{F}^T \underline{P}\ \underline{e} + \underline{M}^{-1} \dot{\underline{x}}] \ . \tag{18}$$

Since the model is stable, a symmetric positive definite matrix $\underline{P}$ can always be found given a symmetric positive definite matrix $\underline{N}$ such that[2],

$$\underline{A}^T_m \underline{P} + \underline{P}\ \underline{A}_m = -\underline{N} \ . \tag{19}$$

Now, in order to guarantee the stability of the system, $\dot{\underline{x}}$ is chosen to make $\dot{V}$ negative semi-definite.

$$\dot{\underline{x}} = -\underline{M}\ \underline{F}^T \underline{P}\ \underline{e} \ , \tag{20}$$

then,

$$\dot{V} = -\underline{e}^T \underline{N}\ \underline{e} \ , \tag{21}$$

which is negative semi-definite. At first glance $\dot{V}$ appears to be negative definite; however, if the variables contained in V are considered it is seen that $\dot{V}$ is only negative semi-definite. Since V contains all of the $\underline{x}$ variables and $\dot{V}$ does not, $\dot{V}$ can equal zero when V is greater than zero. The effect of this is that when the error is zero, parameter misalignment can still exist and the adaptive system will no longer correct. This problem was investigated by Graham[10], who also proposed a solution. No further consideration of the problem will be taken here because, in many applications, forcing the error to zero is sufficient.

Returning to Equation (20) to find the adaptive control laws, and also defining

$$\underline{z} = \underline{P}\ \underline{e}\ ,$$

where (22)

$$z_n = e_1\, p_{1n} + e_2\, p_{2n} + \cdots + e_n p_{nn},$$

and recalling that $\underline{f}_1 = \underline{f}_2 = \cdots = \underline{f}_{n-1} = 0$, $\dot{\underline{x}}$ can now be written as,

$$\dot{\underline{x}} = -\underline{M}\ \underline{f}_n\ z_n \quad . \tag{23}$$

Under the assumption that the plant parameters are constant during adaption, the derivative of $\underline{x}$ from its definition in Equation (10) is seen to be,

$$\dot{\underline{x}} = K_p\ \dot{\underline{k}} \tag{24}$$

where,

$$\underline{k} = [k_1\ k_2\ \cdots\ k_n\ K_c]^T \quad . \tag{25}$$

When Equations (23) and (24) are equated, the following adaptive control laws can be concluded:

$$\dot{\underline{k}} = -\frac{1}{K_p} \underline{M}\, \underline{f}_n\, z_n$$

or,

$$\begin{aligned}
\dot{k}_1 &= -\mu_1\, \theta_{s1}\, z_n/K_p \\
\dot{k}_2 &= -\mu_2\, \theta_{s2}\, z_n/K_p \\
&\vdots \\
\dot{k}_n &= -\mu_n\, \theta_{sn}\, z_n/K_p \\
\dot{k}_{n+1} &= \mu_{n+1}\, r\, z_n/K_p \quad .
\end{aligned} \tag{26}$$

B. Comments

An examination of Equation (26) reveals that in order to implement the adaptive control laws each loop will require a multiplier to produce the product of z_n with the respective θ_{sn} or r, as well as an integrator to obtain the k_i's from their derivatives. If the adaptive control laws were of the form

$$\dot{\underline{k}} = -\underline{M}\, \underline{f}_n \operatorname{sgn} y_n \quad , \tag{27}$$

then the multipliers could be replaced by switches.

The first question is how to arrive at the adaptive control laws of Equation (27) and still guarantee that the system is stable. It was thought that since a quadratic form of Lyapunov function led to a product in the adaptive control laws, a signum function in the Lyapunov function could lead to a signum function in the adaptive control laws. This will be shown for a first order system in Section III, and then for an n^{th} order system in Section IV.

There is a problem working with the signum function because it is discontinuous. Flügge-Lotz[11] has shown that differential equations with discontinuous driving functions may not have solutions. To eliminate this problem, the continuous saturation function, defined in Equation (41) and Figure 3, is used to derive the adaptive control laws. Although the sat function is used in the derivation, the sgn function may still be used to implement the control laws. The justification for this is that as α approaches infinity the saturation function will approach the signum function within any specified error.

III. FIRST ORDER SYSTEM -- NEW DESIGN

A. Choice of V Function

In order to establish the desired adaptive control laws, a simple first order model-referenced adaptive system will be investigated. To simplify the problem, the gain parameter of the model and plant will be the same (i.e., $K_m = K_p$), although this is not a restriction on the system and is only done to simplify the initial derivation. A block diagram of this system is shown in Figure 2. b_1 is considered unknown or to be changing in steps.

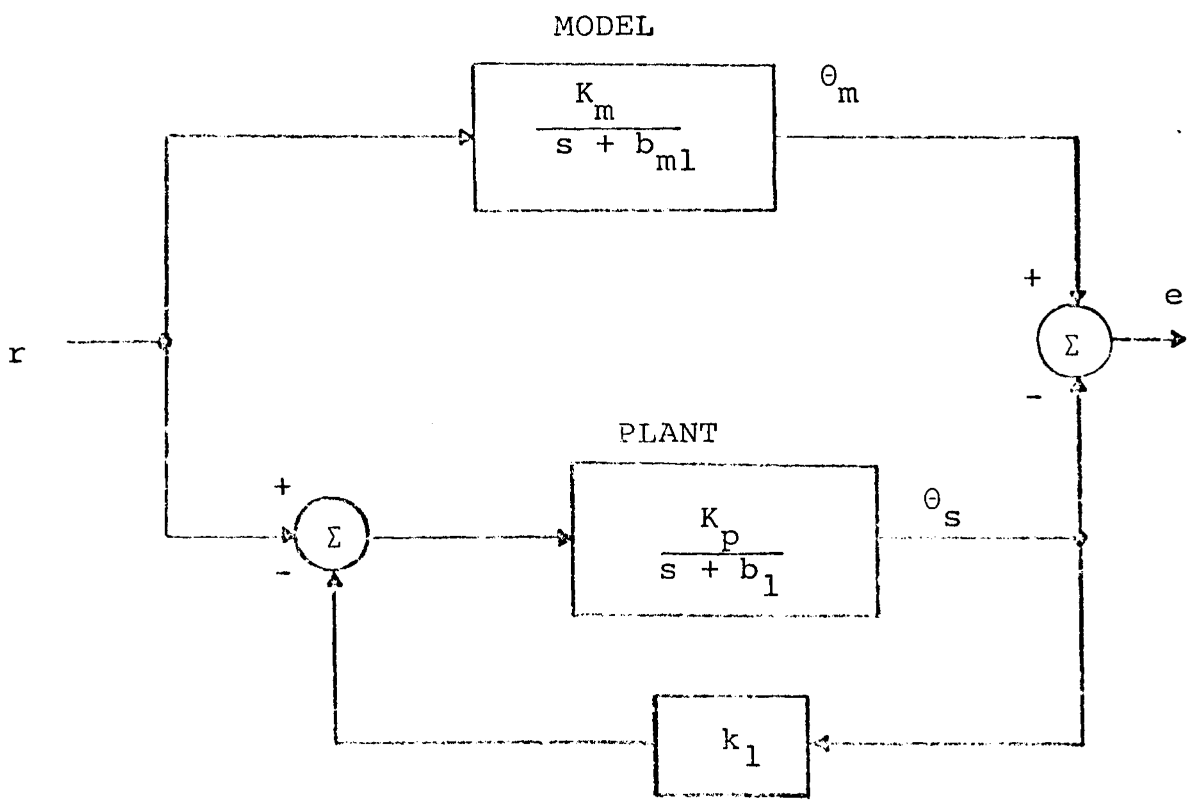

Figure 2. First Order Model-Referenced System

The system equations can be written as follows:

$$\frac{\Theta_m}{r} = \frac{K_m}{s + b_{m1}} \quad , \quad \frac{\Theta_s}{r} = \frac{K_p}{s + b_1 + K_p k_1} \tag{28}$$

and,

$$\dot{\Theta}_m = -b_{m1}\,\Theta_m + K_m r, \quad \dot{\Theta}_s = -(b_1 + K_p k_1)\Theta_s + K_p r \quad . \tag{29}$$

The error equation, with $K_m = K_p$, is then

$$\dot{e} = -b_{m1}\, e + (b_1 + K_p k_1 - b_{m1})\,\Theta_s \quad . \tag{30}$$

If,

$$x_1 \triangleq b_1 + K_p k_1 - b_{m1} \tag{31}$$

then,

$$\dot{e} = -b_{m1}\, e + \Theta_s\, x_1 \quad . \tag{32}$$

One choice of V function is a quadratic form similar to that chosen by Shackcloth[9]

$$V = e^2 + \frac{x_1^2}{K_p \mu_1} \quad . \tag{33}$$

μ_1 is a positive adaptive loop gain and K_p is the gain parameter of the plant which is also positive. V is positive definite. The derivative of V is,

$$\dot{V} = 2\, e\, \dot{e} + 2\, x_1\, \dot{x}_1/K_p\, \mu_1 \quad . \tag{34}$$

Substituting for $\dot{e}$ from Equation (32),

$$\dot{V} = -2\, b_{m1}\, e^2 + 2\, e\, x_1\, \Theta_s + 2\, x_1\, \dot{x}_1/K_p \mu_1 \quad . \tag{35}$$

If the following assignment is made for $\dot{x}_1$,

$$\dot{x}_1 = -K_p\, \mu_1\, e\, \Theta_s, \tag{36}$$

then,

$$\dot{V} = -2\ b_{m1}\ e^2, \tag{37}$$

which is negative semi-definite since $b_{m1} > 0$ for a stable model. The system is therefore stable and e goes to zero.

Under the assumption that b_1 is constant during adaption,

$$\dot{x}_1 = K_p \dot{k}_1 \quad . \tag{38}$$

When Equations (36) and (38) are equated, the adaptive control law is found to be,

$$\dot{k}_1 = -\mu_1\ e\ \Theta_s \quad . \tag{39}$$

It is seen that in order to implement this adaptive control law a multiplier is needed to obtain the product of e and Θ_s.

A V function which may lend itself to a sat function in the adaptive control law is,

$$V = \int_0^e \mathrm{sat}\ \alpha\ \tau\ d\tau + \frac{x_1^2}{2\mu_1\ K_p} \quad . \tag{40}$$

The sat function is defined as,

$$\mathrm{sat}\ \alpha\ e = \begin{cases} \alpha\ e & \text{for } |\alpha e| < 1 \\ 1 & \text{for } |\alpha e| > 1 \end{cases} , \tag{41}$$

and is illustrated in Figure 3.

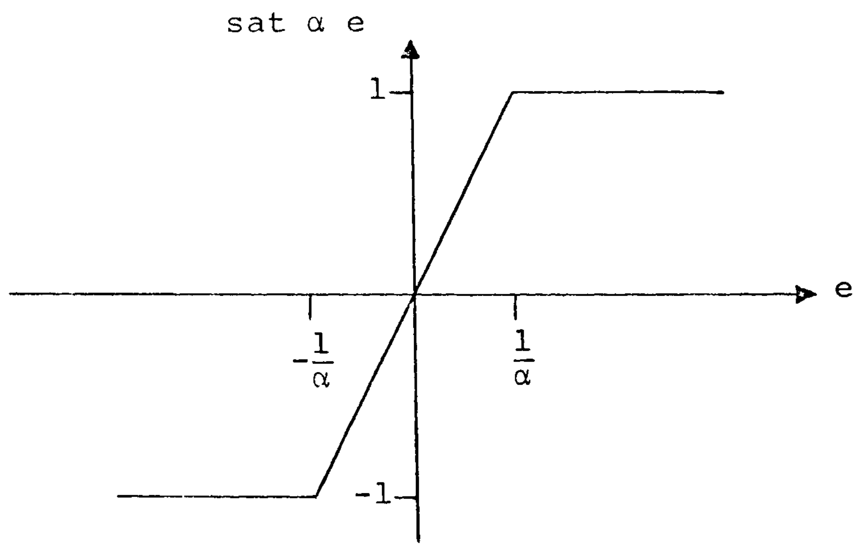

Figure 3. The sat Function

The derivative of V is,

$$\dot{V} = \dot{e} \text{ sat } \alpha \text{ e} + \frac{x_1 \dot{x}_1}{K_p \mu_1} \quad . \tag{42}$$

When Equation (32) is substituted into Equation (42),

$$\dot{V} = -b_{m1} \text{ e sat } \alpha \text{ e} + x_1 \Theta_s \text{ sat } \alpha \text{ e} + \frac{x_1 \dot{x}_1}{K_p \mu_1} \quad . \tag{43}$$

If the following choice is made,

$$\dot{x}_1 = -K_p \; \mu_1 \; \Theta_s \text{ sat } \alpha \text{ e} \quad , \tag{44}$$

then,

$$\dot{V} = -b_{m1} \text{ e sat } \alpha \text{ e} \quad . \tag{45}$$

Since $\dot{V}$ is negative semi-definite, the system is again guaranteed to be stable and e goes to zero. However, the new adaptive control law is,

$$\dot{k}_1 = -\mu_1 \; \Theta_s \text{ sat } \alpha \text{ e} \quad . \tag{46}$$

As α is made to approach infinity, sat α e approaches sgn e. Now, in the limit Equation (40) becomes,

$$V = \int_0^e \text{sgn}\ \tau\ d\ \tau + x_1^2/2\mu_1 K_p \tag{47}$$

and, in the limit, Equation (43) becomes,

$$\dot{V} = -b_{m1}\ e\ \text{sgn}\ e + x_1\ \Theta_s\ \text{sgn}\ e + \frac{x_1 \dot{x}_1}{K_p \mu_1}\ . \tag{48}$$

If,

$$\dot{x}_1 = -K_p\ \mu_1\ \Theta_s\ \text{sgn}\ e \tag{49}$$

then,

$$\dot{V} = -b_{m1}\ e\ \text{sgn}\ e \tag{50}$$

which is negative semi-definite again. The adaptive control law is now,

$$\dot{k}_1 = -\mu_1\ \Theta_s\ \text{sgn}\ e\quad . \tag{51}$$

Equation (51) is the desired adaptive control law since it can be implemented with a switch.

B. Analog Simulation

To investigate the operation of a system using the control law of Equation (46), the system shown in Figure 4 was simulated on an Electronic Associates, Inc., TR-48 analog computer. The simulation diagram is in the Appendix.

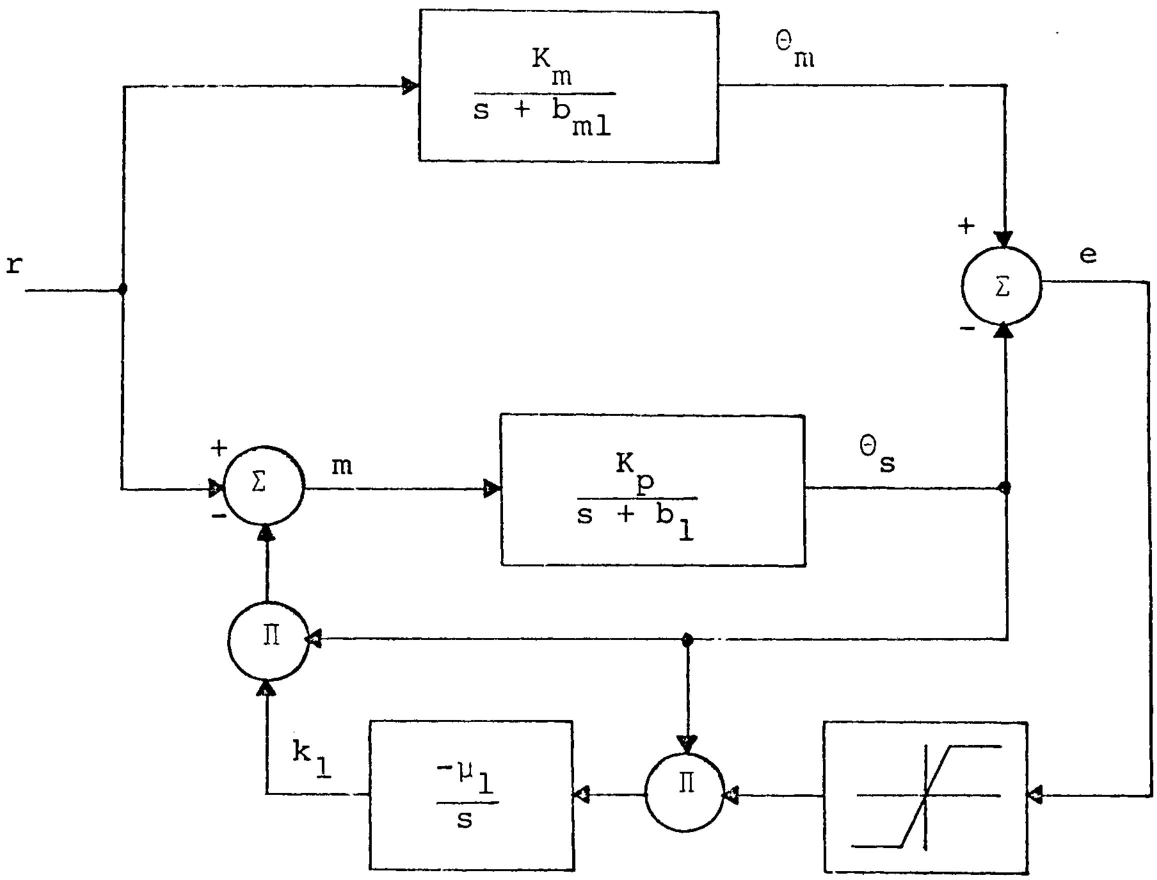

Figure 4. First Order System With sat Function in the Adaptive Control Law

The following parameters were arbitrarily chosen for the initial simulation:

$$\begin{array}{lll} K_m = K_p = 2, & b_{m1} = 1 \ , & b_1 = 11 \ , \\ \mu_1 = 1 \ , & \alpha = 10 \ , & r = \sin t, \\ \theta_m(0) = 0, & \theta_s(0) = 0, & k_1(0) = 0 \ . \end{array} \tag{52}$$

The results of the simulation are shown in Figure 5. It is seen that the controlled plant responds as the model does with the sat function in the adaptive control law. The adaption is slow, however, and this problem will be examined in the next section.

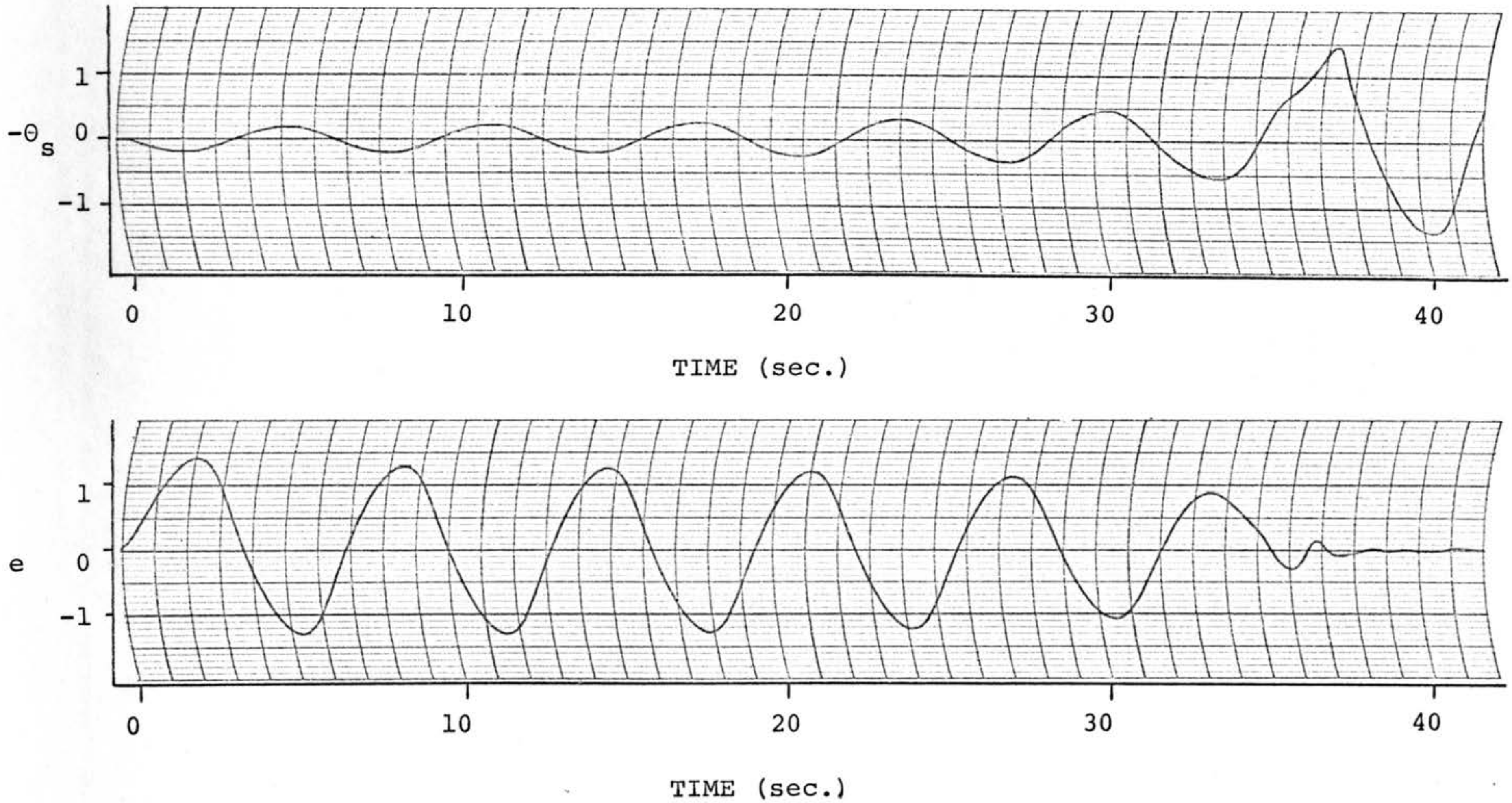

Figure 5. θ_s, e, k_1 with sat Function Adaptive Control Law

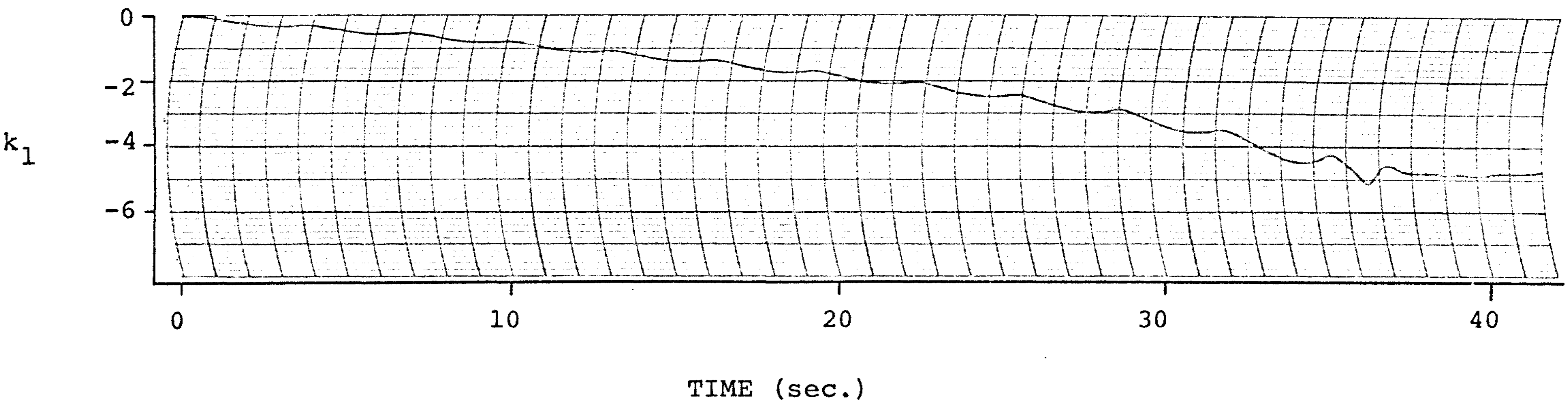

Figure 5. Continued

As pointed out in the introduction, the discontinuous property of the signum function necessitated the use of the saturation function in derivation of the adaptive control laws. However, to implement the saturation function requires the use of a multiplier, hence there is no simplification of the hardware with the adaptive control law of Equation (46). The improvement in hardware comes when the signum function is used. To examine what effect the signum function in the adaptive control law has on the system performance, the system of Figure 6 was simulated using the adaptive control law of Equation (51). The same parameters and initial conditions were used as in Equation (52).

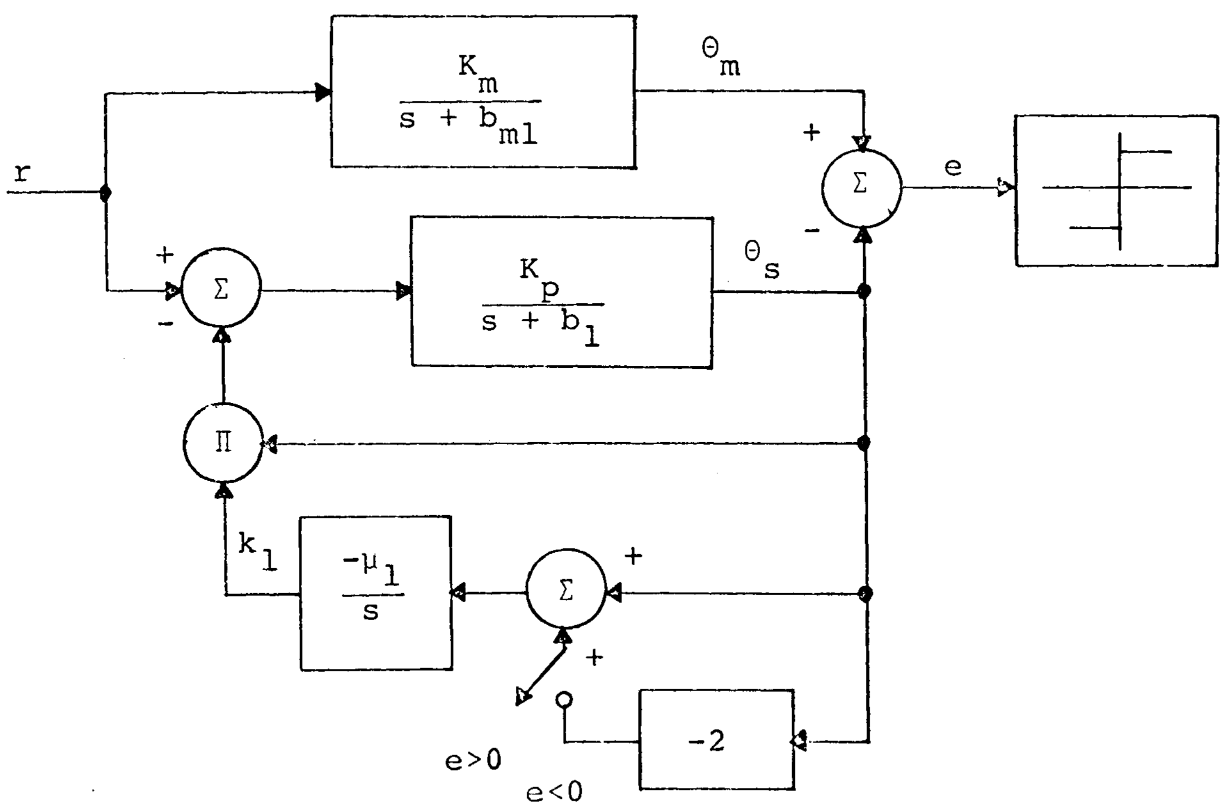

Figure 6. First Order System with sgn Function Adaptive Control Law

The analog simulation diagram is in the Appendix, and the results are shown in Figure 7. It appears that in this case the signum function is better than the saturation function because adaption is much faster. A closer examination of the plant output Θ_s indicates that there is a high frequency signal present after adaption that was not present when the sat function was used. The amplitude of this high frequency signal, however, is small. This problem will be discussed in more detail in the next section.

As a comparison of the new design using the adaptive control law of Equation (51) and the existing design using the law of Equation (39), the system of Figure 6 was simulated again with the same parameters and initial conditions but with the comparator replaced by a multiplier. The other input to the multiplier was the plant output. The simulation diagram is in the Appendix and the results are shown in Figure 8. To make a fair comparison, much further investigation would be necessary. However, on the basis of this simple system it appears that the new design is superior in at least two respects, hardware and speed of adaption.

C. Bounds on Peak Error and Adaption Time

The system of the previous section using the sgn function in the adaptive control law does the job of adapting, but several improvements can be made. The first

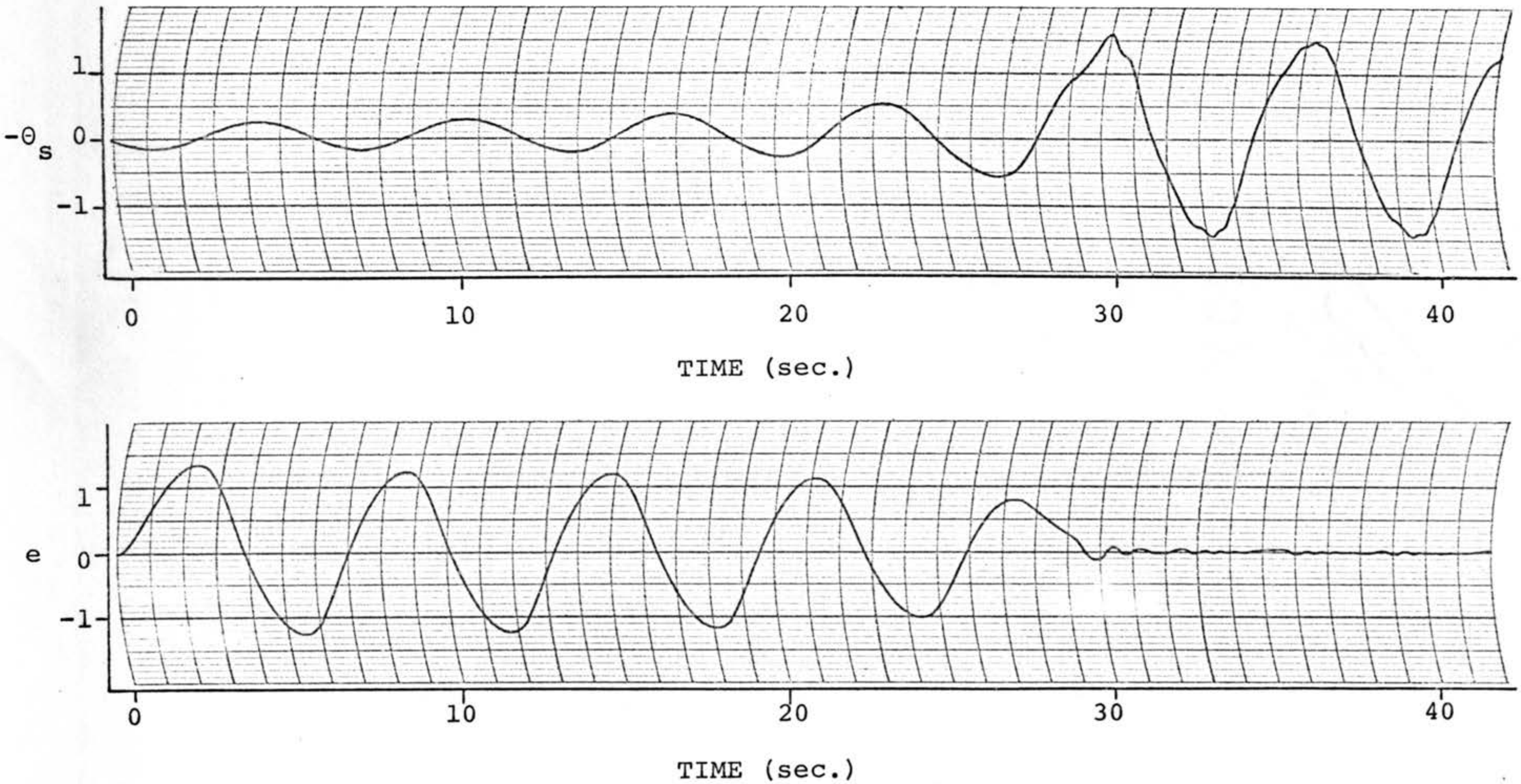

Figure 7. Θ_s, e, k_1 with sgn Function Adaptive Control Law

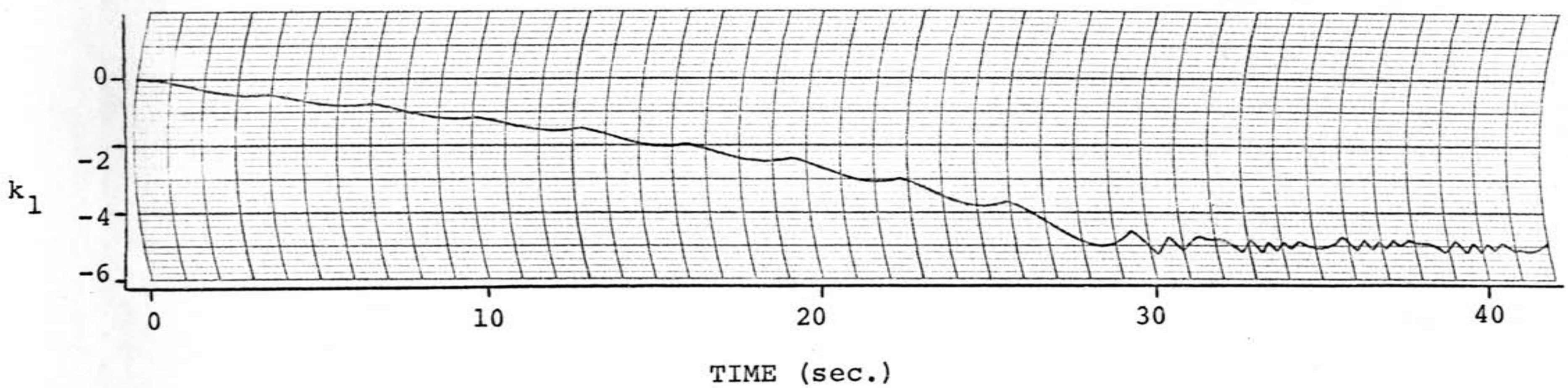

Figure 7. Continued

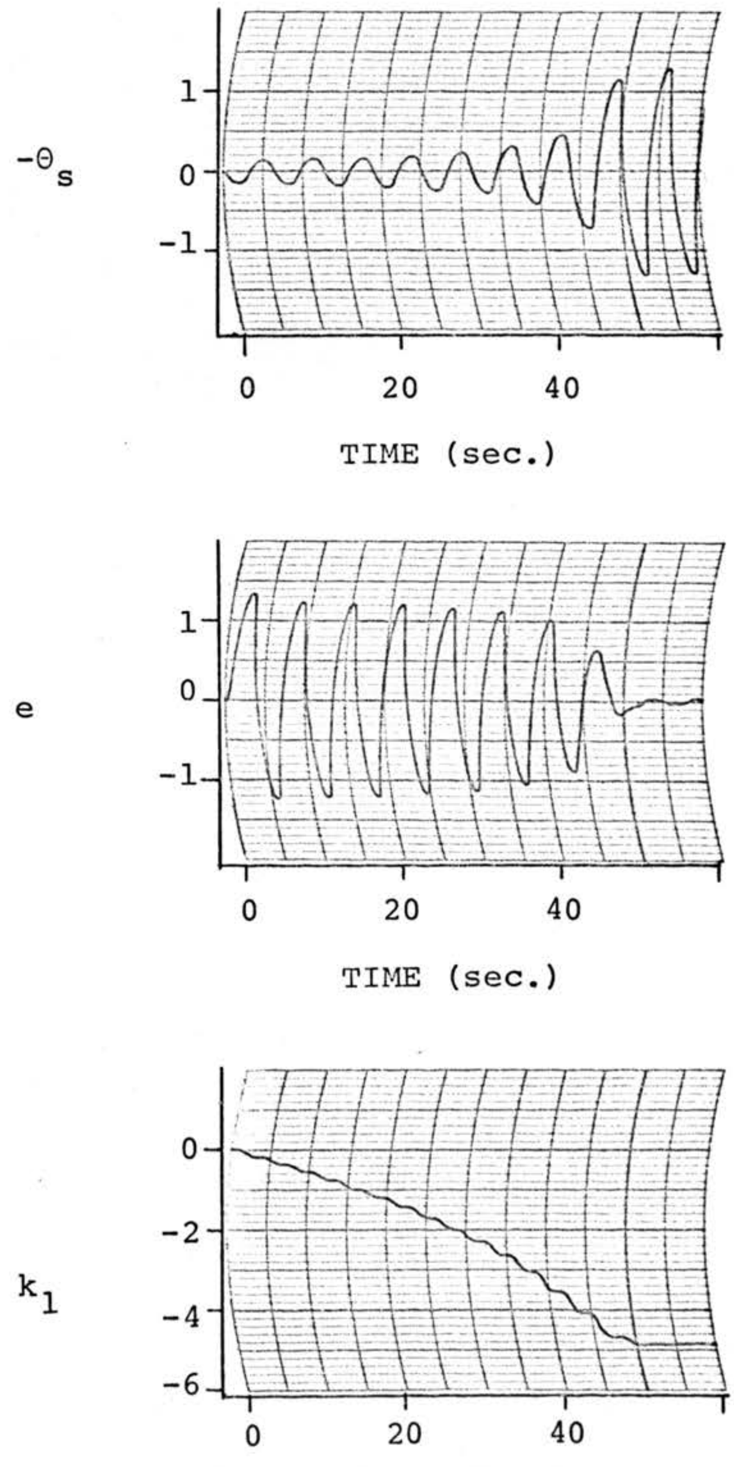

Figure 8. Θ_s, e, k_1 with Product Adaptive Control Law

feature investigated was the adaption time. This is the time that it takes for the magnitude of the error to stay below a stated value. The results in Figure 7 indicate that the time to reduce the error to less than 0.1 volt is 30 seconds. This may be unsatisfactory.

The first impulse was to examine the figure of merit[12] normally associated with the Lyapunov function,

$$N = \frac{-\dot{V}}{V} \quad . \tag{53}$$

An upper bound on the system time constant is $1/N_{MIN}$. For the system with adaptive control law (51),

$$N = \frac{b_{m1}\ e \ \mathrm{sgn}\ e}{\int_0^e \mathrm{sgn}\ \tau \ d\tau + \frac{x_1^2}{2\mu_1 K_p}} \quad . \tag{54}$$

From Equation (54) it is seen that the minimum value of N is zero when e is zero, so a bound on the system time constant cannot be set by the figure of merit method.

However, an upper bound on the error after a step change in parameter occurs, can be set. This can be seen by examining the V function. Since the derivative of V is negative semi-definite,

$$V(0) \geq V(t) \qquad \text{for} \quad t > 0 \quad . \tag{55}$$

When the integral of Equation (47) is evaluated, the resulting equation for V is,

$$V = |e| + x_1^2/2\mu_1\ K_p \quad . \tag{56}$$

After Equation (31) is substituted for x_1 and V is evaluated at $t = 0$, the result is,

$$V(0) = |e(0)| + \frac{[b_1 + k_1(0)K_p - b_{m1}]^2}{2\,\mu_1\,K_p} \quad , \tag{57}$$

and if $k_1(0) = e(0) = 0$,

$$V(0) = \frac{[b_1 - b_{m1}]^2}{2\mu_1 K_p} \quad . \tag{58}$$

An examination of Equations (55) and (56) shows that the absolute error will never be greater than V and that V is at its maximum at $t = 0$. Now, for time greater than zero, if the entire value of V were due to the error term, the maximum absolute error would be $\frac{[b_1 - b_{m1}]^2}{2\mu_1 K_p}$. Hence Equation (58) is an upper bound on the absolute error. The actual peak error will probably be less than this. It is also concluded that by increasing μ_1, the upper bound on the absolute error is decreased.

Several experiments were performed with different values of μ_1. In particular, the system in Figure 6 was simulated again using a value of $\mu_1 = 100$. From Equation (58) the upper bound on the error is .25 as compared to 25 with $\mu_1 = 1$. The results of this simulation are shown in Figure 9. The peak error is less than 0.2. It is also seen that there is now an increase in frequency of the high frequency component of Θ_s. This is because the switch used

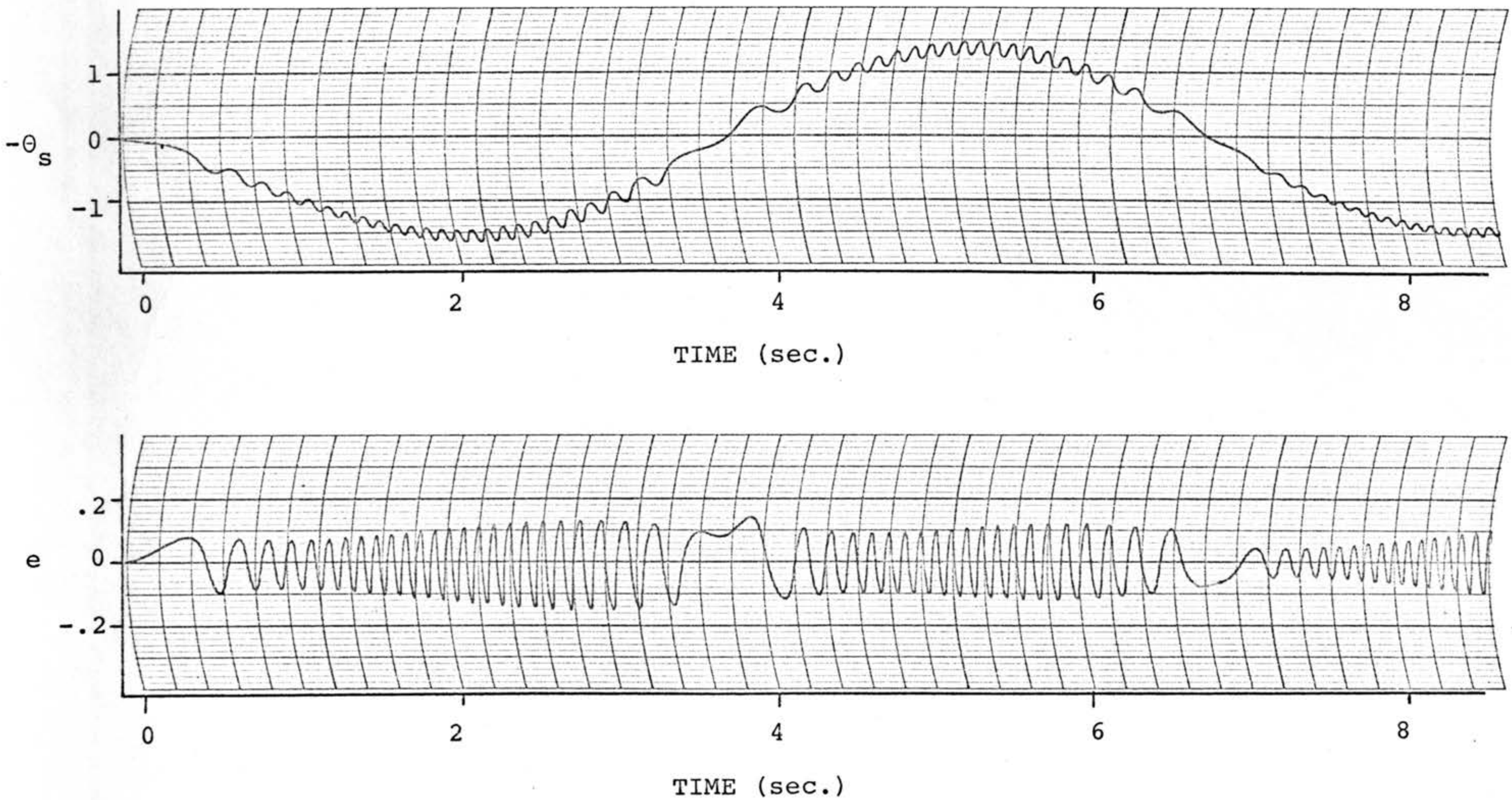

Figure 9. Θ_s, e, k_1 with sgn Function Adaptive Control Law and μ_1 = 100

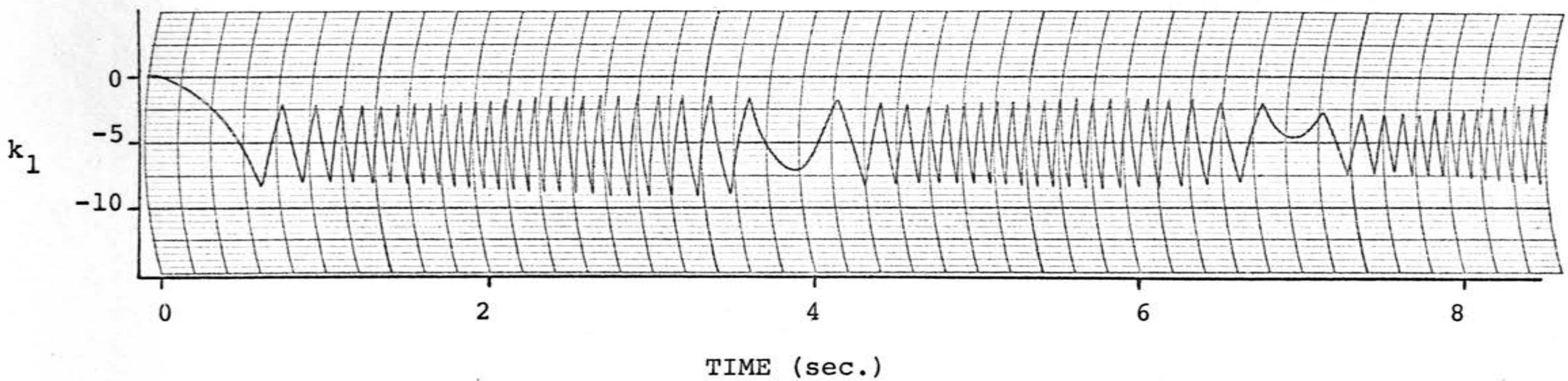

Figure 9. Continued

to implement sgn ϵ was oscillating at a faster rate about e equals zero. It should be pointed out that there is no equilibrium point for the system of Figure 6. If these high frequencies are detrimental to the system, one possible remedy would be to use a relay with deadzone in it. Then, when the error became less than a certain amount, the adaptive loop gain would be zero. Hence an equilibrium region would exist. This engineering aspect was not investigated in this thesis.

D. Time Varying Parameters

Up to this point, the only cases covered were the plant with unknown parameters, and the plant with parameters varying by steps. What about the plant with randomly varying parameters? Certainly it would be nice to include this case also.

The only difference in the equations derived in Section III A is the introduction of an additional term in the derivative of x_1 for time varying parameters.

Now, with K_p constant and b_1 varying,

$$\dot{x}_1 = \dot{b}_1 + K_p \dot{k}_1 \quad . \tag{59}$$

This reflects back into $\dot{V}$, as shown below. Using the adaptive control law of Equation (46), $\dot{x}_1$ now becomes,

$$\dot{x}_1 = \dot{b}_1 - K_p \mu_1 \Theta_s \text{ sat } \alpha e \quad . \tag{60}$$

From (43), $\dot{V}$ is,

$$\dot{V} = -b_{m1} e \text{ sat } \alpha e + \frac{\dot{b}_1 x_1}{K_p \mu_1} \quad . \tag{61}$$

Unfortunately, $\dot{V}$ is no longer negative semi-definite. There is a disturbance term of $\frac{\dot{b}_1 x_1}{K_p \mu_1}$ which is of undetermined sign. This term can be made smaller by increasing μ_1. However, it must be realized that increasing μ_1 also decreases the effect parameter misalignment has on V. Therefore, the system is not satisfactory for time varying parameters as it stands now.

There is a small modification of the system which will increase the ability to handle time varying parameters. The improvement was suggested by Phillipson[7] to decrease the oscillatory nature of the system. It also makes $\dot{V}$ more negative and therefore tends to cancel out the disturbance term of (61).

The modification suggested by Phillipson is to add an additional feedback path including the derivative of k_1 as seen in Figure 10. β_1 is a constant which controls the amount of derivate feedback.

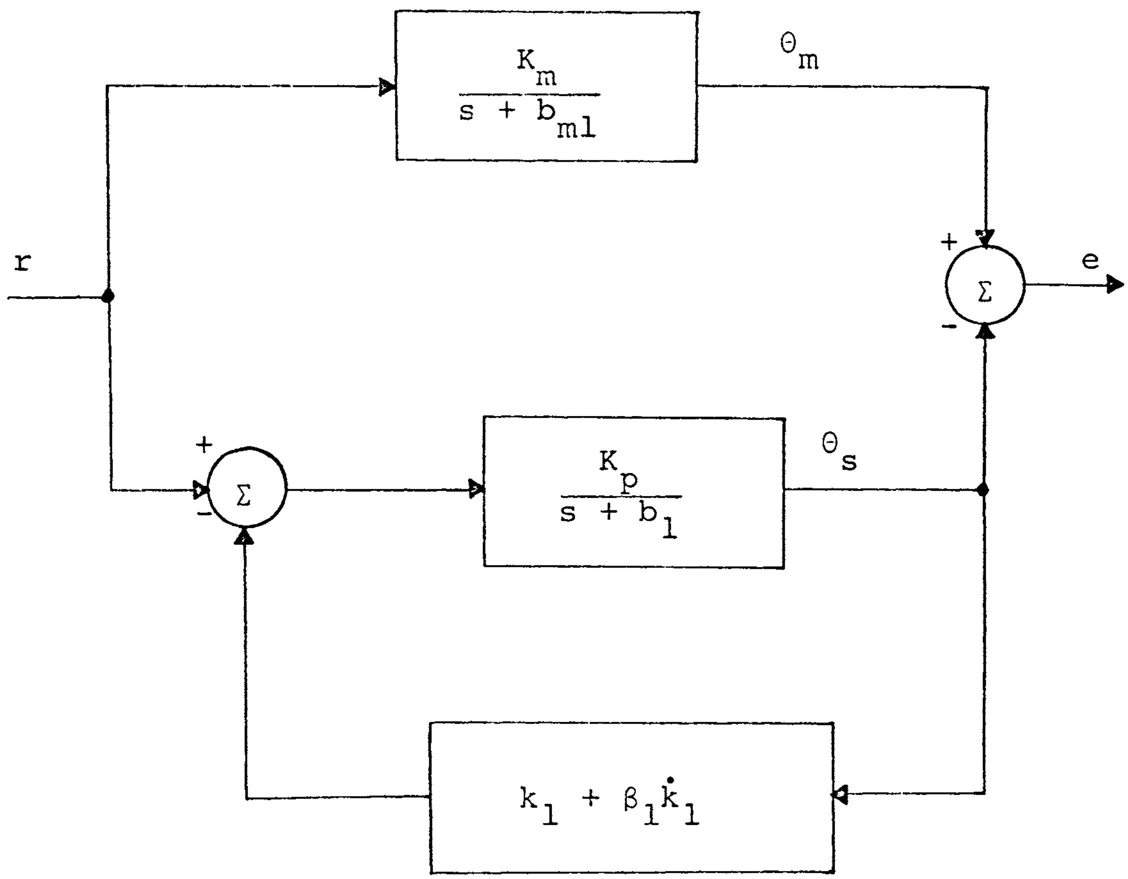

Figure 10. System with k_1 Feedback Added

The system equations are now:

$$\dot{\Theta}_m = -b_{m1}\Theta_m + K_m r, \quad \dot{\Theta}_s = -(b_1 + K_p k_1)\Theta_s + K_p r - K_p \beta_1 \dot{k}_1 \Theta_s \quad . \tag{62}$$

If $K_p = K_m$ again,

$$\dot{e} = -b_{m1} e + \Theta_s x_1 + K_p \beta_1 \dot{k}_1 \Theta_s \quad . \tag{63}$$

When V is chosen as in Equation (40),

$$\dot{V} = \dot{e} \text{ sat } \alpha \; e + x_1 \; \dot{x}_1 / K_p \; \mu_1 \quad . \tag{64}$$

After substitution of (63) into (64), the result is,

$$\dot{V} = -b_{m1}\ e\ \text{sat}\ \alpha\ e + \Theta_s x_1\ \text{sat}\ \alpha\ e + K_p \beta_1 \dot{k}_1 \Theta_s\ \text{sat}\ \alpha\ e + \frac{x_1\ \dot{x}_1}{K_p\ \mu_1} \tag{65}$$

If the adaptive control law of Equation (46) is implemented, $\dot{x}_1$ will again be as in Equation (60) and thus,

$$\dot{V} = -b_{m1}\ e\ \text{sat}\ \alpha\ e - K_p \beta_1 \mu_1 \Theta_s^2\ \text{sat}^2\ \alpha\ e + \frac{\dot{b}_1\ x_1}{K_p\ \mu_1}\quad . \tag{66}$$

A comparison of Equations (61) and (66) indicates that $\dot{V}$ is more negative in Equation (66). Without the derivative of k_1 in the feedback path, a small error would tend to make the disturbance term the sign determining factor. However, with the additional feedback, the second term of (66), which can be made very large by increasing μ_1, will usually be much larger than the disturbance term. Although $\dot{V}$ is not negative semi-definite, confidence that it will not be positive is certainly increased. The only possibility of a problem is when Θ_s remains very small. Under most circumstances, this will not be the case.

To verify the ability to adapt to time varying parameters the system of Figure 10 was simulated using the control law of Equation (51). Only a slight modification on the simulation diagram for Figure 6 was necessary. An additional amplifier and potentiometer were added to supply the derivative of k_1 feedback. The following parameters were used:

$$K_p = 2, \quad b_{m1} = 1, \quad \beta_1 = 0.1$$

$$\mu_1 = 100, \quad \Theta_s(0) = 0, \quad r = \sin t, \tag{67}$$

$$\Theta_m(0) = 0, \quad k_1(0) = 0 \quad \text{and}$$

b_1 is a random variable with Gaussian distribution, mean = 8, standard deviation = 3.14, bandwidth = 1 radian/second.

The results in Figure 11 indicate excellent adaption. Also, the high frequency signal is no longer present in Θ_s; however, it does show up in the feedback term $k_1 + \beta_1\dot{k}_1$. The affects of this high frequency signal should definitely be taken into consideration when designing a system.

E. Modification for the Gain Parameter Varying

From a system in which the model and the plant have the same gain parameter, it is a simple step to add another loop to compensate for the varying or unknown additional parameter of the plant. The system in Figure 12 can compensate for changes in K_p by changing K_c.

To derive the adaptive control laws, the system equations are examined.

$$\dot{\Theta}_s = (K_c + \beta_2\dot{K}_c)K_p r - (b_1 + (k_1 + \beta_1\dot{k}_1)K_p)\Theta_s \quad , \tag{68}$$

$$\dot{\Theta}_m = K_m r - b_{m1}\Theta_m \quad , \tag{69}$$

and,

$$\dot{e} = -b_{m1}e + (K_m - K_c K_p)r - \beta_2\dot{K}_c K_p r + (b_1 + k_1 K_p - b_{m1})\Theta_s$$
$$+ \beta_1\dot{k}_1 K_p\Theta_s \quad . \tag{70}$$

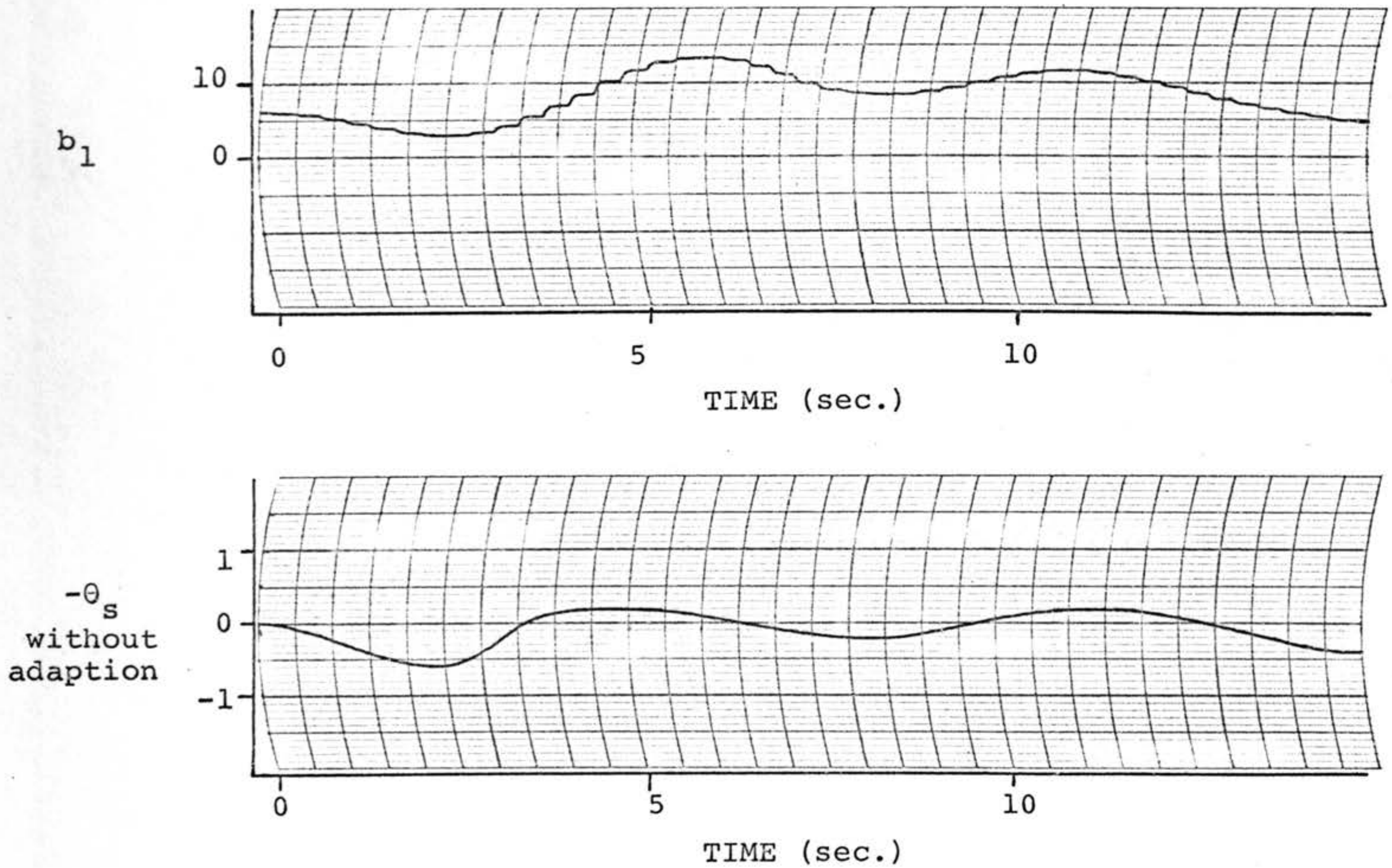

Figure 11. b_1, Θ_s Without Adaptive Mechanism, Θ_s With Adaptive Mechanism, e With Adaptive Mechanism, for Randomly Varying b_1

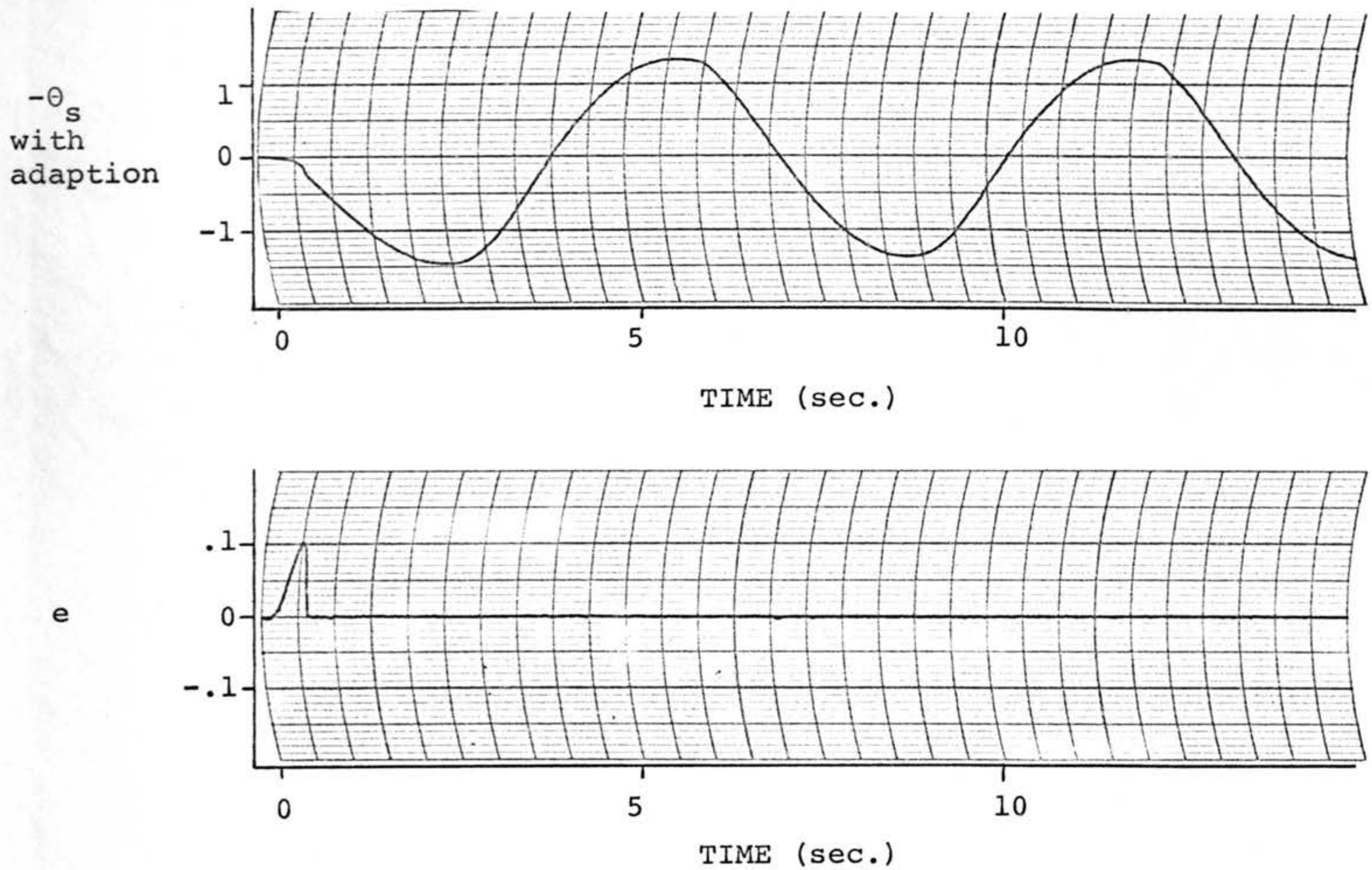

Figure 11. Continued

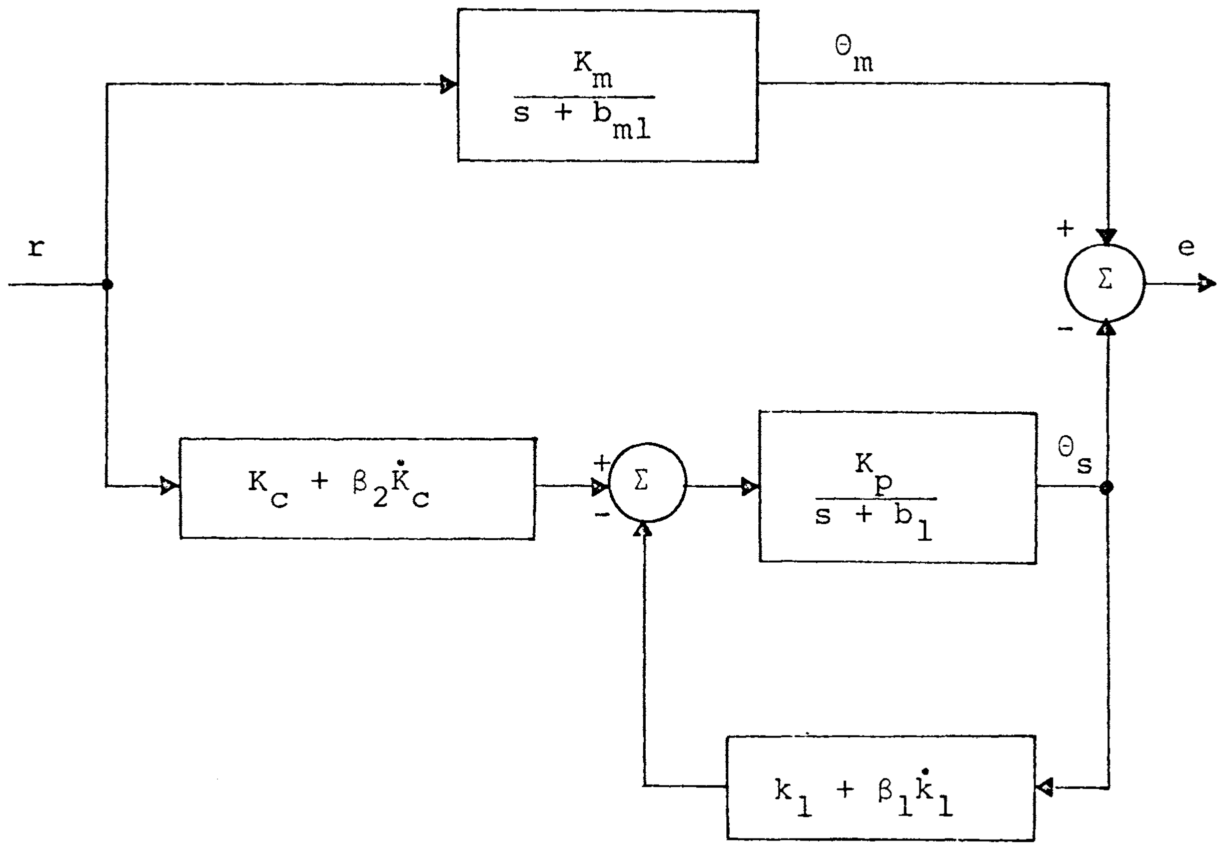

Figure 12. System with Two Varying Parameters

The following definitions are made,

$$x_1 = b_1 + k_1\, K_p - b_{m1} \tag{71}$$

and

$$x_2 = K_m - K_c\, K_p \quad . \tag{72}$$

Now,

$$\dot{e} = -b_{m1}e + x_2 r - \beta_2 \dot{K}_c K_p r + x_1 \Theta_s + \beta_1 \dot{k}_1 K_p \Theta_s \quad . \tag{73}$$

A choice of V function similar to Equation (40) is made but with a term to include the variation in gain parameter.

$$V = \int_0^e \text{sat}\ \alpha\ \tau\ d\ \tau \ + \frac{x_1^2}{2\mu_1 K_p} + \frac{x_2^2}{2\mu_2 K_p} \quad . \tag{74}$$

K_p, although varying, is assumed to be always positive, hence V is positive definite.

$$\dot{V} = -b_{m1}\ e\ \text{sat}\ \alpha\ e + x_2\ r\ \text{sat}\ \alpha\ e - \beta_2 \dot{K}_c K_p\ r\ \text{sat}\ \alpha\ e$$
$$+ x_1 \Theta_s\ \text{sat}\ \alpha\ e + \beta_1 \dot{k}_1 K_p \Theta_s\ \text{sat}\ \alpha\ e + \frac{x_1 \dot{x}_1}{\mu_1 K_p} - \frac{x_1^2 \dot{K}_p}{2\mu_1 K_p^2}$$
$$+ \frac{x_2 \dot{x}_2}{\mu_2 K_p} - \frac{x_2^2 \dot{K}_p}{2\mu_2 K_p^2} \tag{75}$$

In order to make $\dot{V}$ as negative as possible, adaptive control laws similar to Equation (46) are chosen,

$$\dot{k}_1 = -\mu_1 \Theta_s\ \text{sat}\ \alpha\ e \tag{76}$$

and

$$\dot{K}_c = \mu_2\ r\ \text{sat}\ \alpha\ e \quad . \tag{77}$$

Now, if both K_p and b_1 are time varying,

$$\dot{x}_1 = \dot{b}_1 + \dot{k}_1 K_p + k_1 \dot{K}_p \tag{78}$$

and,

$$\dot{x}_2 = -\dot{K}_c K_p - K_c \dot{K}_p \quad . \tag{79}$$

When Equation (76) is substituted into Equation (78) and (77) is substituted into (79), the result is, with $k_1(0) = K_c(0) = 0$,

$$\dot{x}_1 = \dot{b}_1 - \mu_1 K_p \Theta_s \text{ sat } \alpha \text{ e} - \mu_1 \dot{K}_p \int_0^t \Theta_s(\tau) \text{ sat } \alpha \text{ e}(\tau) \, d\tau \tag{80}$$

and,

$$\dot{x}_2 = -\mu_2 K_p r \text{ sat } \alpha \text{ e} - \mu_2 \dot{K}_p \int_0^t r(\tau) \text{ sat } \alpha \text{ e}(\tau) \, d\tau \tag{81}$$

$\dot{V}$ is now,

$$\begin{aligned} \dot{V} = &-b_{m1} \text{ e sat } \alpha \text{ e} - \mu_2 \beta_2 K_p r^2 \text{ sat}^2 \alpha \text{ e} - \mu_1 \beta_1 K_p \Theta_s^2 \text{ sat}^2 \alpha \text{ e} \\ &+ \frac{x_1 \dot{b}_1}{\mu_1 K_p} - \frac{x_1^2 \dot{K}_p}{2\mu_1 K_p^2} - \frac{x_2^2 \dot{K}_p}{2\mu_2 K_p^2} - \frac{\dot{K}_p x_1}{K_p} \int_0^t \Theta_s(\tau) \text{ sat } \alpha \text{ e}(\tau) \, d\tau \\ &- \frac{\dot{K}_p x_2}{K_p} \int_0^t r(\tau) \text{ sat } \alpha \text{ e}(\tau) \, d\tau \, . \end{aligned} \tag{82}$$

When the parameters K_p and b_1 are constant, the last five terms in Equation (82) are zero and $\dot{V}$ is negative semi-definite. However, when K_p and b_1 are varying the last five terms in Equation (82) are sign undetermined disturbances on $\dot{V}$. Although the negative terms can be made very large by increasing the adaptive loop gains, stability cannot be guaranteed while the plant parameters are changing.

Some simulations were made with the system of Figure 12, with both b_1 and K_p time varying. The controlled plant again responded the same as the model. Since this topic could be made into a separate report, the investigations were not pursued any further.

F. Comments

In Section III a first order model referenced system was examined. The adaptive control law proposed in Equation (51) was shown to be an effective gain adjustment criteria for controlling a plant with unknown or step changing parameters. The new adaptive control law has an advantage over the previous adaptive control law which requires a multiplier, since it can be implemented by a switch. On the basis of the first order system, adaption is faster using the new adaptive control law than using the old one. It was shown that an upper bound on the error can be set to any value by adjusting the adaptive loop gain. However, high gains increased the frequency of the spurious signal present in the output.

Randomly varying parameters are another problem. Addition of the derivative feedback path increases the probability of system stability by making the derivative of V more negative. Unfortunately, stability still cannot be guaranteed while the parameters are changing.

IV. N^{TH} ORDER SYSTEM -- NEW DESIGN

A. Choice of V Function

Of primary importance in deriving the adaptive control laws for the n^{th} order system is the proper choice of V function. As stated in the introduction, there is no formal procedure for choosing the V function. A trial and error procedure was used until the desired adaptive control laws were obtained. The matrix notation used in Section IIA will be used throughout this section. In addition, some new matrices will be defined. The sat function will be used in the derivations for the reasons mentioned previously.

An n x n matrix $\underline{Q}$ is first defined as follows:

$$\underline{Q} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & & & \\ \vdots & & & \vdots \\ q_{n1} & & \cdots & q_{nn} \end{bmatrix} = \begin{bmatrix} \underline{q}_1^T \\ \underline{q}_2^T \\ \vdots \\ \underline{q}_n^T \end{bmatrix} . \tag{83}$$

Also an n column matrix is defined,

$$\underline{\text{sat}}\ \alpha\ \underline{Q}\ \underline{e} = \begin{bmatrix} \text{sat}\ \alpha\ \underline{q}_1^T\ \underline{e} \\ \text{sat}\ \alpha\ \underline{q}_2^T\ \underline{e} \\ \vdots \\ \text{sat}\ \alpha\ \underline{q}_n^T\ \underline{e} \end{bmatrix} . \tag{84}$$

Finally, define a vector function

$$\underline{w}(\underline{Q}\ \underline{e}) = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} \quad ; \tag{85}$$

where,

$$w_i = \int_0^{\underline{q}_i^T \underline{e}} \text{sat}\ \alpha\ \tau\ d\ \tau\ . \tag{86}$$

Now, the following choice of V function is made:

$$V = \sum_{i=1}^{n} w_i + \frac{1}{2K_p} \underline{x}^T \underline{M}^{-1} \underline{x} \quad ; \tag{87}$$

where $\underline{x}$ is defined in Equation (10) and $\underline{M}$ is defined in Equation (14). V is positive definite if $\underline{Q}$ is non-singular. Differentiation of V gives,

$$\dot{V} = (\underline{\text{sat}}\ \alpha\ \underline{Q}\ \underline{e})^T \underline{Q}\ \dot{\underline{e}} + \frac{1}{2K_p} \dot{\underline{x}}^T \underline{M}^{-1} \underline{x} + \frac{1}{2K_p} \underline{x}^T \underline{M}^{-1} \dot{\underline{x}}\ . \tag{88}$$

Substitution of $\dot{\underline{e}}$ from Equation (12) yields,

$$\dot{V} = (\underline{\text{sat}}\ \alpha\ \underline{Q}\ \underline{e})^T \underline{Q}\ \underline{A}_m\ \underline{e} + (\underline{\text{sat}}\ \alpha\ \underline{Q}\ \underline{e})^T \underline{Q}\ \underline{F}\ \underline{x} + \frac{1}{K_p} \dot{\underline{x}}^T \underline{M}^{-1} \underline{x}\ . \tag{89}$$

The objective again is to choose the adaptive control laws so that $\dot{V}$ is negative semi-definite. A procedure is used similar to that of the scalar case in Equation (43).

The second term of (89) is cancelled by the third term by choosing,

$$\dot{\underline{x}} = -K_p \, \underline{M} \, \underline{F}^T \, \underline{Q}^T \, \underline{\text{sat}} \, \alpha \, \underline{Q} \, \underline{e} \quad . \tag{90}$$

Now,

$$\dot{V} = (\underline{\text{sat}} \, \alpha \, \underline{Q} \, \underline{e})^T \, \underline{Q} \, \underline{A}_m \, \underline{e} \quad . \tag{91}$$

$\dot{V}$ is not necessarily negative semi-definite as it was for the scalar case. It may be negative semi-definite for certain values of $\underline{Q}$. The problem now is to determine the proper $\underline{Q}$, if it exists.

One way for a function of the form,

$$\dot{V} = y \text{ sat } \alpha \, x \tag{92}$$

to be negative definite is for the following relationship to hold,

$$y = c \, x \quad ; \tag{93}$$

where, c is a negative constant. Applying this criteria to Equation (91) implies

$$\underline{C} \, \underline{Q} \, \underline{e} = \underline{Q} \, \underline{A}_m \, \underline{e} \quad . \tag{94}$$

Equation (94) is certainly true if,

$$\underline{C} \, \underline{Q} = \underline{Q} \, \underline{A}_m \quad . \tag{95}$$

If $\underline{C}$ is a diagonal matrix of negative real constants, then (91) will be negative semi-definite. Since $\underline{Q}$ is nonsingular, Equation (95) can be post multiplied by $\underline{Q}^{-1}$ and then,

$$\underline{C} = \underline{Q} \, \underline{A}_m \cdot \underline{Q}^{-1} \quad . \tag{96}$$

The problem is now reduced to diagonalizing the model matrix $\underline{A}_m$ by a similarilty transform. It can be shown[12] that if the eigenvalues of a matrix $\underline{A}_m$ are distinct, then it can be tranformed into a diagonal matrix $\underline{\Lambda}$ as follows:

$$\underline{\Lambda} = \underline{T}^{-1} \underline{A}_m \underline{T} \quad . \tag{97}$$

$\underline{\Lambda}$ is a diagonal matrix of the eigenvalues λ_i, and $\underline{T}$ is the Vandermonde matrix as follows,

$$\underline{T} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_1^2 & & \lambda_n^2 \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & & \lambda_n^{n-1} \end{bmatrix} \quad . \tag{98}$$

If the model is stable and all eigenvalues of $\underline{A}_m$ are real, $\underline{\Lambda}$ will be a diagonal matrix of negative real constants and the conditions for negative semi-definiteness of (91) will be satisfied.

The required value of $\underline{Q}$ is

$$\underline{Q} = \underline{T}^{-1} \quad . \tag{99}$$

$\underline{Q}$ will be a real matrix since all λ_i are real. The derivative of V is now,

$$\dot{V} = (\underline{\text{sat}}\ \alpha\ \underline{T}^{-1}\ \underline{e})^T\ \underline{\Lambda}\ \underline{T}^{-1}\ \underline{e} \quad . \tag{100}$$

Since (100) is negative semi-definite, the model-referenced system is stable and the error goes to zero.

The adaptive control laws can now be determined from Equation (90). It is assumed that the plant parameters are constant during adaption as in Equation (24). Equating

Equation (24) and Equation (90) yields,

$$\underline{\dot{k}} = -\underline{M}\ \underline{F}^T\ \underline{Q}^T\ \underline{\text{sat}}\ \alpha\ \underline{Q}\ \underline{e} \quad . \tag{101}$$

In summary, the results of this section show that the scalar case of control law (46) can be extended to the vector case of control law (101). The model is restricted to be stable and to have real distinct eigenvalues. These restrictions will be discussed again in Section IV C.

B. A Second Order Example

To provide more insight into a system other than first order, the system of Figure 13 was examined. Plant parameters b_1 and b_2 are considered to be unknown or subject to unknown step changes.

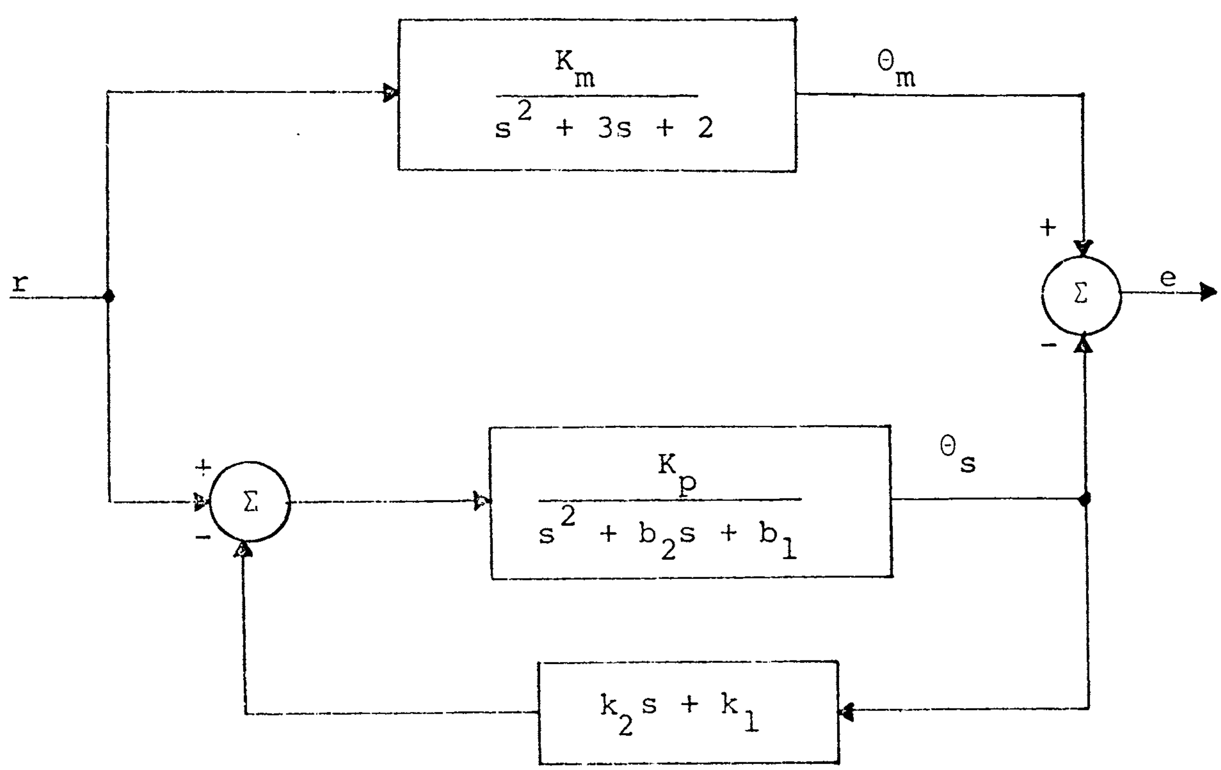

Figure 13. Second Order System

The system equations are,

$$\dot{\underline{\theta}}_m = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{\theta}_m + \begin{bmatrix} 0 \\ K_m \end{bmatrix} r \tag{102}$$

and,

$$\dot{\underline{\theta}}_s = \begin{bmatrix} 0 & 1 \\ -(b_1+K_pk_1) & -(b_2+K_pk_2) \end{bmatrix} \underline{\theta}_s + \begin{bmatrix} 0 \\ K_p \end{bmatrix} r \quad . \tag{103}$$

For simplification, K_p and K_m are again made equal. The error equation is then,

$$\dot{\underline{e}} = \dot{\underline{\theta}}_m - \dot{\underline{\theta}}_s = \underline{A}_m \underline{e} + (\underline{A}_m - \underline{A}_s) \underline{\theta}_s \quad , \tag{104}$$

or,

$$\dot{\underline{e}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{e} + \begin{bmatrix} 0 & 0 \\ b_1 + K_pk_1 - 2 & b_2 + K_pk_2 - 3 \end{bmatrix} \underline{\theta}_s \quad . \tag{105}$$

If the following definitions are made,

$$x_1 = b_1 + K_pk_1 - 2 \quad \text{and} \quad x_2 = b_2 + K_pk_2 - 3 \tag{106}$$

then,

$$\dot{\underline{e}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{e} + \begin{bmatrix} 0 & 0 \\ \theta_{s1} & \theta_{s2} \end{bmatrix} \underline{x} \quad . \tag{107}$$

To derive the adaptive control laws, V is chosen as in Equation (87),

$$V = \int_0^{y_1} \text{sat } \alpha \tau \, d\tau + \int_0^{y_2} \text{sat } \alpha \tau \, d\tau + \frac{1}{2K_p} \underline{x}^T \underline{M}^{-1} \underline{x} \quad ; \qquad (108)$$

where,

$$y_1 = \underline{q}_1^T \underline{e} \ , \quad \text{and} \quad y_2 = \underline{q}_2^T \underline{e} \ . \qquad (109)$$

If $\underline{q}_1 \neq b \, \underline{q}_2$, V is positive definite. To determine $\underline{Q}$ the eigenvalues of $\underline{A}_m$ are evaluated, and

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2 \ . \qquad (110)$$

Now, from Equation (98)

$$\underline{T} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \qquad (111)$$

and,

$$\underline{Q} = \underline{T}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} . \qquad (112)$$

Also,

$$y_1 = [2 \quad 1] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 2e_1 + e_2 \ . \qquad (113)$$

and

$$y_2 = [-1 \quad -1] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = -e_1 - e_2 \ . \qquad (114)$$

Returning to the V function, its derivative can now be taken,

$$\dot{V} = \dot{y}_1 \text{ sat } \alpha\, y_1 + \dot{y}_2 \text{ sat } \alpha\, y_2 + \frac{1}{K_p} \underline{\dot{x}}^T \underline{M}^{-1} \underline{x} \quad . \tag{115}$$

From (113) and (114)

$$\dot{y}_1 = 2\dot{e}_1 + \dot{e}_2 \tag{116}$$

and,

$$\dot{y}_2 = -\dot{e}_1 - \dot{e}_2 \quad . \tag{117}$$

Equation (107) shows that,

$$\dot{e}_1 = e_2 \tag{118}$$

and,

$$\dot{e}_2 = -2e_1 - 3e_2 + x_1\theta_{s1} + x_2\theta_{s2} \quad . \tag{119}$$

After substitution of (118) and (119) into (116) and (117), and (116) and (117) into (115), V can be written,

$$\begin{aligned} \dot{V} = & -y_1 \text{ sat } \alpha\, y_1 - 2y_2 \text{ sat } \alpha\, y_2 \\ & + x_1\, \theta_{s1}(\text{sat } \alpha\, y_1 - \text{sat } \alpha\, y_2) \\ & + x_2\, \theta_{s2}(\text{sat } \alpha\, y_1 - \text{sat } \alpha\, y_2) \\ & + \frac{1}{K_p} \underline{\dot{x}}^T \underline{M}^{-1} \underline{x} \quad . \end{aligned} \tag{120}$$

To make $\dot{V}$ negative semi-definite, $\underline{\dot{x}}$ is chosen according to Equation (90),

$$\underline{\dot{x}} = -K_p \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \begin{bmatrix} 0 & \theta_{s1} \\ 0 & \theta_{s2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \text{sat } \alpha\, y_1 \\ \text{sat } \alpha\, y_2 \end{bmatrix} \tag{121}$$

or,

$$\underline{\dot{x}} = \begin{bmatrix} -\mu_1 K_p \Theta_{s1}(\text{sat } \alpha\, y_1 - \text{sat } \alpha\, y_2) \\ -\mu_2 K_p \Theta_{s2}(\text{sat } \alpha\, y_1 - \text{sat } \alpha\, y_2) \end{bmatrix} . \tag{122}$$

Substitution of (122) back into (120) yields,

$$\dot{V} = -y_1 \text{ sat } \alpha\, y_1 - 2y_2 \text{ sat } \alpha\, y_2 \quad . \tag{123}$$

$\dot{V}$ is negative semi-definite as expected. The following adaptive control laws can be concluded

$$\dot{k}_1 = -\mu_1 \Theta_{s1}(\text{sat } \alpha\, y_1 - \text{sat } \alpha\, y_2) \tag{124}$$

and,

$$\dot{k}_2 = -\mu_2 \Theta_{s2}(\text{sat } \alpha\, y_1 - \text{sat } \alpha\, y_2) \quad . \tag{125}$$

A further simplification of the adaptive control laws can be made by using the fact that the derivative of the V function does not have to be negative definite in $\underline{e}$ to guarantee asymptotic stability. $\dot{V}$ can be negative semi-definite[2] in $\underline{e}$ as long as the system does not have any other equilibrium point except when V = 0.

For the system of Figure 13, V can be chosen,

$$V = \int_0^{y_1} \text{sat } \tau \, d\, \tau + \frac{1}{2K_p} \underline{x}^T \underline{M}^{-1} \underline{x} \quad . \tag{126}$$

The adaptive control laws then become,

$$\dot{k}_1 = -\mu_1 \Theta_{s1} \text{ sat } \alpha\, y_1 \tag{127}$$

and,

$$\dot{k}_2 = -\mu_2 \Theta_{s2} \text{ sat } \alpha\, y_1 \quad . \tag{128}$$

Now,

$$\dot{V} = -y_1 \text{ sat } \alpha \, y_1 \quad . \tag{129}$$

$\dot{V}$ is only negative semi-definite since

$$y_1 = 2e_1 + e_2 = 0 \tag{130}$$

is a condition in which $\dot{V}$ is zero and the error may not be zero. However, solving (13) yields,

$$2e + \dot{e} = 0 \tag{131}$$

and

$$e = c \, e^{-2t} \quad , \tag{132}$$

indicating that the error will still decay to zero. Hence, the adaptive control laws of (127) and (128) can be used with a reduction in hardware.

Before simulation, the sat function was replaced by the sgn function as before, the final adaptive control laws becomes

$$\dot{k}_1 = -\mu_1 \, \Theta_s \, \text{sgn}(2e + \dot{e})$$

and (133)

$$\dot{k}_2 = -\mu_2 \, \dot{\Theta}_s \, \text{sgn}(2e + \dot{e}) \quad .$$

The system of Figure 13 was simulated using the adaptive control laws of Equation (133) and the following parameters,

$$K_p = 2 \, , \quad \mu_1 = 100 \, , \quad \mu_2 = 100 \, ,$$

$$\Theta_m(0) = 0 \, , \quad \Theta_s(0) = 0, \quad k_1(0) = 0, \tag{134}$$

$$k_2(0) = 0 \, , \quad r = \sin t, \quad b_1 = 20 \, ,$$

$b_2 = 5 + s(t)$, $s(t)$ = square wave of amplitude ±3 frequency of 1 radian/second .

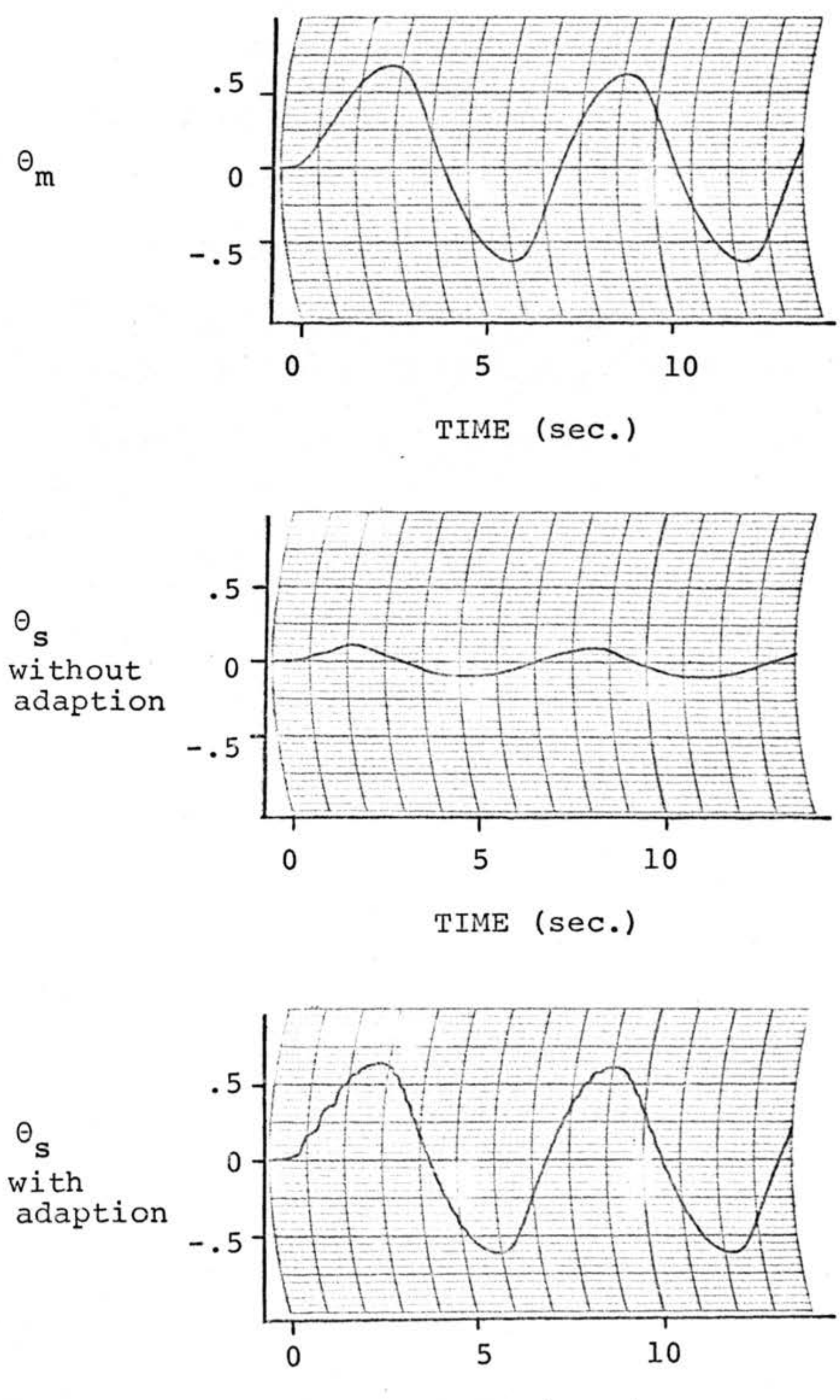

Figure 14. Θ_m, Θ_s Without Adaptive Mechanism, Θ_s With Adaptive Mechanism, for Second Order System

The derivative of k_1 and the derivative of k_2 were added to the feedback as discussed in Section III D. The simulation diagram is in the Appendix and the results are shown in Figure 14.

A comparison of the output of the plant without adaption and the output with adaption, indicates a rapid adaption with very little error. The results definitely confirm that the adaptive control laws derived in Section IV A are useful.

C. Restrictions on the Model

The derivation of the adaptive control laws in Section IV A imposed restrictions on the model so that $\dot{V}$ would be negative semi-definite. The restrictions are repeated here for emphasis. The model must satisfy the following requirements:

1. it must be stable,
2. all eigenvalues must be distinct, and
3. all eigenvalues must be real.

The stability of the model is required so that the matrix $\underline{\Lambda}$, of Equation (97), will have all negative numbers on its diagonal. If this were not true, then $\dot{V}$ would contain a positive definite term. The model will most likely be stable in any practical system, therefore, this is really not a limitation.

Next, the restriction of distinct eigenvalues was imposed so that a transformation was guaranteed to exist which would yield a diagonal matrix $\underline{\Lambda}$ in Equation (97). If $\underline{A}_m$ does not have distinct eigenvalues, then it may not be similar to a diagonal matrix[12]. This, however, does not imply that $\dot{V}$ of Equation (91) will not be negative semi-definite, but it does imply that all cases of non-diagonal matrices must be checked to see if $\dot{V}$ could possibly be positive. This was not done because it was thought that in most cases the model would have distinct eigenvalues. Further investigations should include a study of the effects of non-distinct eigenvalues on $\dot{V}$.

So far, none of the restrictions have been a serious limitation. However, there will undoubtedly be many cases where a model with complex roots will be desired. The choice of $\underline{Q}$ made in Equation (99) does not work if the model has complex roots. An examination of $\underline{T}$ in Equation (98) indicates that complex eigenvalues would make $\underline{T}$ complex and in turn $\underline{Q}$ complex. The adaptive control laws of (101) cannot be implemented in this case because they require the saturation function with complex argument which is undefined. This situation can be partially resolved by making an additional transformation.

Instead of transforming $\underline{A}_m$ to a diagonal matrix $\underline{\Lambda}$ as in (97), it will be now transformed into a matrix $\underline{J}$ of the form[12],

$$\underline{J} = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \cdot & & & & \\ & & \sigma & \omega & & \\ & & -\omega & \sigma & & \\ & & & & \cdot & \\ 0 & & & & & \lambda_n \end{bmatrix} \quad ; \tag{135}$$

where σ is the real part of the complex eigenvalue and ω is the imaginary part. This transformation can be made as follows:

$$\underline{J} = \underline{S}^{-1}\, \underline{T}^{-1}\, \underline{A}_m\, \underline{T}\, \underline{S} \quad . \tag{136}$$

$\underline{T}$ is the same as defined in (98) and $\underline{S}$ is,

$$\underline{S} = \begin{bmatrix} 1 & & & & & 0 \\ & \cdot & & & & \\ & & \frac{1}{2} & -\frac{j}{2} & & \\ & & \frac{1}{2} & \frac{j}{2} & & \\ & & & & \cdot & \\ 0 & & & & & 1 \end{bmatrix} \quad . \tag{137}$$

The significance is that not only is $\underline{J}$ real but so is $\underline{T}\ \underline{S}$. Now $\underline{Q}$ can be chosen as follows:

$$\underline{Q} = \underline{S}^{-1}\, \underline{T}^{-1} \quad . \tag{138}$$

The adaptive control laws will be the same as in Equation (101) except with the $\underline{Q}$ of (138), which is real.

The form of $\dot{V}$ will now be,

$$\dot{V} = (\underline{\text{sat}}\ \alpha\ \underline{Q}\ \underline{e})^T\ \underline{J}\ \underline{Q}\ \underline{e} \, . \tag{139}$$

All terms of $\dot{V}$ will be negative definite as before, except for those corresponding to the off diagonal elements of $\underline{J}$

due to the complex eigenvalues. A closer examination of (139) is necessary.

Defining $\underline{y} = \underline{Q}\ \underline{e}$ and multiplying out $\underline{J}\ \dot{\underline{y}}$ yields,

$$\dot{V} = [\text{sat}\alpha y_1 \cdots \text{sat}\alpha y_m\ \text{sat}\alpha y_{m+1} \cdots \text{sat}\alpha y_n] \begin{bmatrix} \lambda_1 y_1 \\ \cdot \\ \cdot \\ \cdot \\ \sigma y_m + \omega y_{m+1} \\ -\omega y_m + \sigma y_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n y_n \end{bmatrix} . \tag{141}$$

The terms of $\dot{V}$ due to complex roots are,

$$\sigma y_m\ \text{sat}\alpha y_m + \omega y_{m+1}\ \text{sat}\alpha y_m$$

and, (142)

$$-\omega y_m\ \text{sat}\alpha y_{m+1} + \sigma y_{m+1}\ \text{sat}\alpha y_{m+1} \quad .$$

Since the model is stable, σ is negative; hence, the first and last term of (142) are both negative definite. The second and third terms can be of either sign, but they are of different sign from each other. Therefore,

$$\omega y_{m+1}\ \text{sat}\alpha y_m - \omega y_m\ \text{sat}\alpha y_{m+1} \leq \omega \max\ (|y_{m+1}|, |y_m|) \ . \tag{143}$$

The largest possible positive contribution to $\dot{V}$ is $\omega[\max(|y_{m+1}|,\ |y_m|)]$. If $|\sigma| > \omega$, then the first or fourth terms of (142) will always be more negative than the positive terms. Hence, $\dot{V}$ will again be negative semi-definite, the system will be stable and e will go to zero.

Thus the restriction of the model having only real roots has been reduced to the model having only real roots and complex roots whose real part is greater than its imaginary part. This will permit much more flexibility in the choice of a model. The same technique can be applied to models with more than one set of complex poles.

V. SUMMARY AND RECOMMENDATIONS

Some comment on the use of Lyapunov's direct method as a design technique is in order. Much of the time spent developing this new design was in trial and error. The desired results were known but the Lyapunov function was not. Many different V functions were tried without success. Finally, an adequate one was chosen. Although trial and error was involved in arriving at the design procedure, from this point on there is no trial and error involved if signum function adaptive control laws are desired. The Lyapunov direct method definitely has merit as a design technique. Quite often it will be the only method available.

The design presented in this thesis provides a method for the control of all of the parameters of a controlled plant. Explicit identification of the plant dynamics is unnecessary since a model is used as a reference for adjusting the parameters. Each parameter is adjusted by means of a feedback loop. The form of the feedback loop is determined by the adaptive control law for that loop. Each adaptive control law can be implemented by a switch and an integrator. The main advantage of the system is that it is guaranteed to be asymptotically stable when the parameters are not varying. Hence, this design is most applicable to systems in which the plant parameters are constant but cannot be measured. It is also applicable to systems in which the plant parameters are changing in steps since the time when the parameters are varying is small. There is a very high probability that the system will be stable for slowly varying parameters, but strictly speaking, stability can only be guaranteed when the parameters are constant.

When choosing a design for a particular system, the advantages must be weighed against the disadvantages. There are some disadvantages inherent in the design presented in this thesis. First of all, if the model has complex poles the real part of the complex poles must be greater than the imaginary part. Second, a number of derivatives of the error and the plant output must be generated. The problems associated with taking derivatives in a physical system are well known.

Hopefully, some of the limitations can be removed, or at least lessened, as a result of further research. Some suggestions follow.

1. Extend the design to a system whose model has arbitrary poles. Possibly a different transformation of the model matrix, or a different V function will produce this result.
2. Reduce the adaptive control laws to logic form. For the first order case, this would be $\dot{k}_1 = \mu_1 \operatorname{sgn} \Theta_s \operatorname{sgn} e$. The product of two signum functions can be implemented by an exclusive-or gate.

3. Investigate the engineering aspects of this design in more detail. The use of switches with deadzone and/or hysteresis should be investigated.

4. Extend the design to systems where the model and the plant are of different order. There may be a problem in this extension due to purely algebraic control loops.

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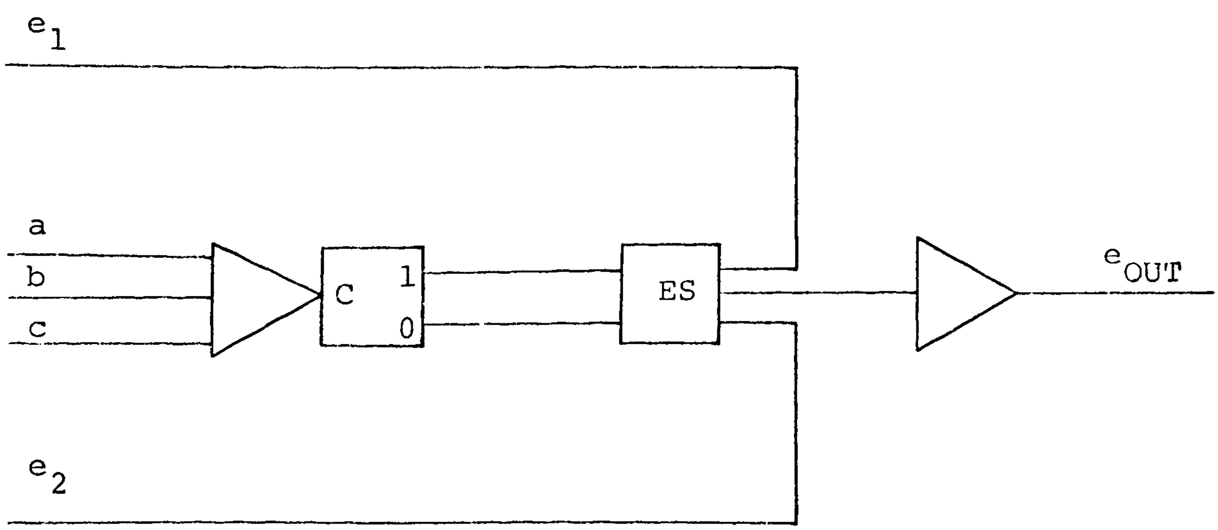
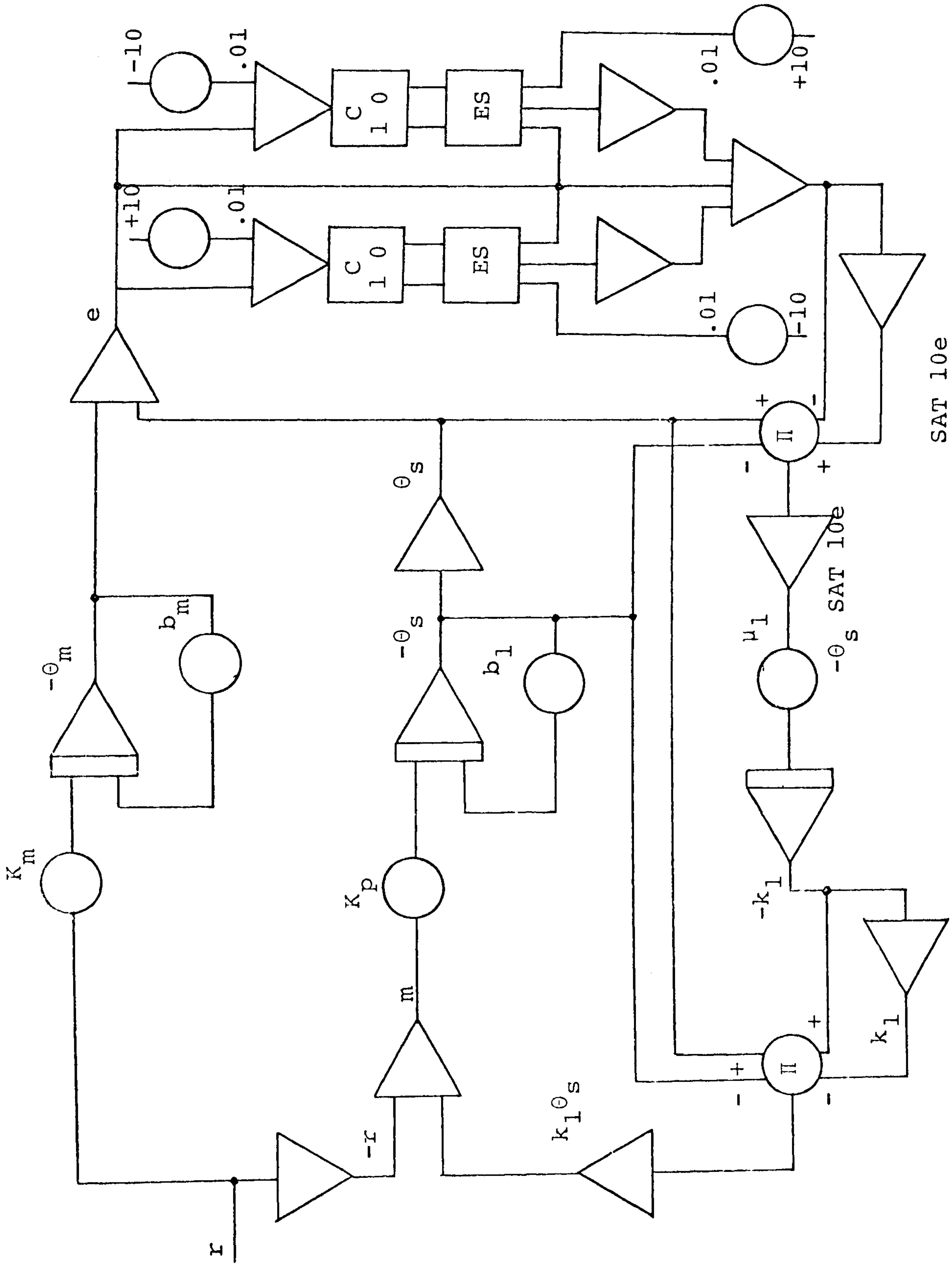
APPENDIX

1. Comparator and Electronic Switch Diagram

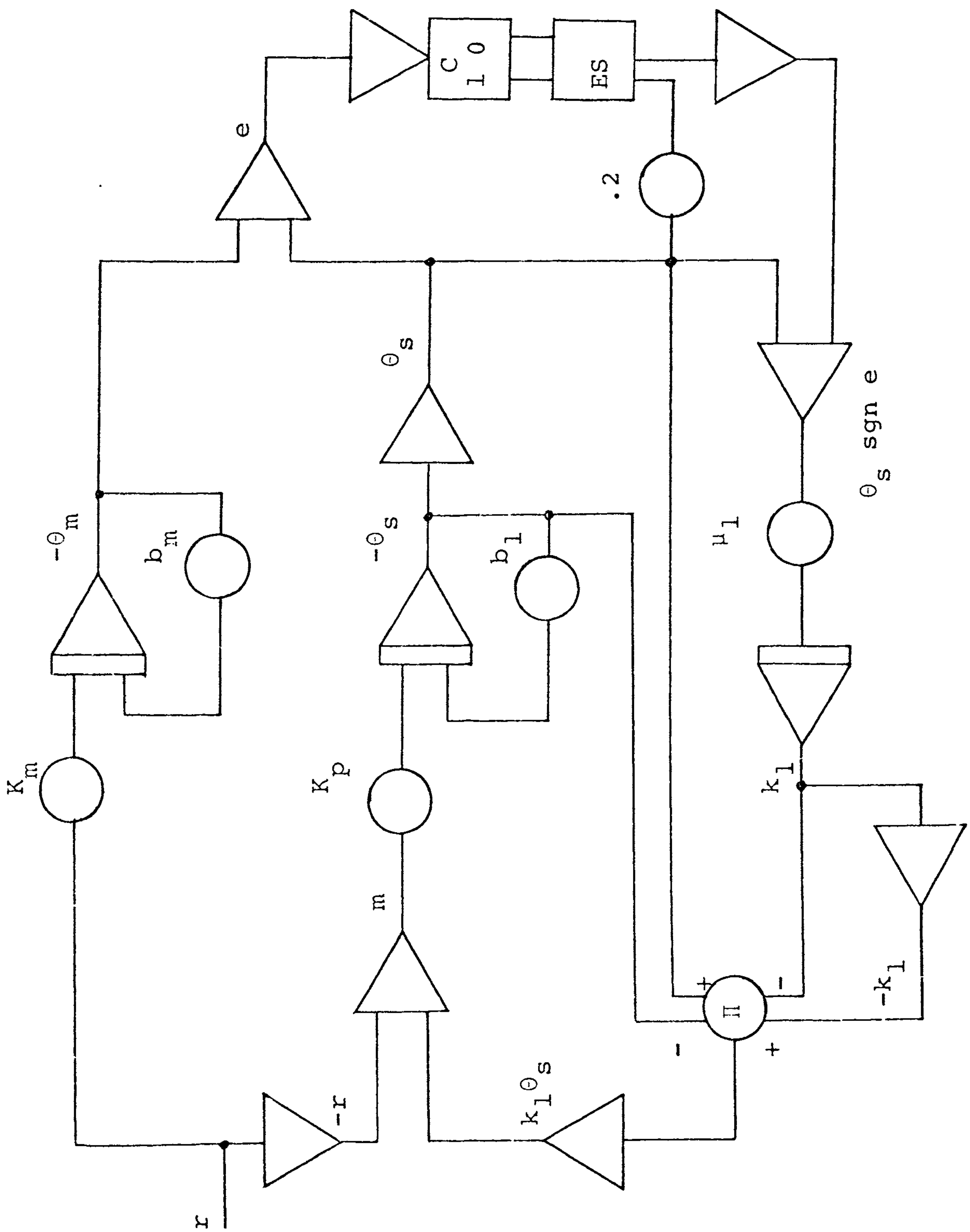


$$e_{OUT} = \begin{cases} -e_1, & a + b + c > 0 \\ -e_2, & a + b + c < 0 \end{cases}$$

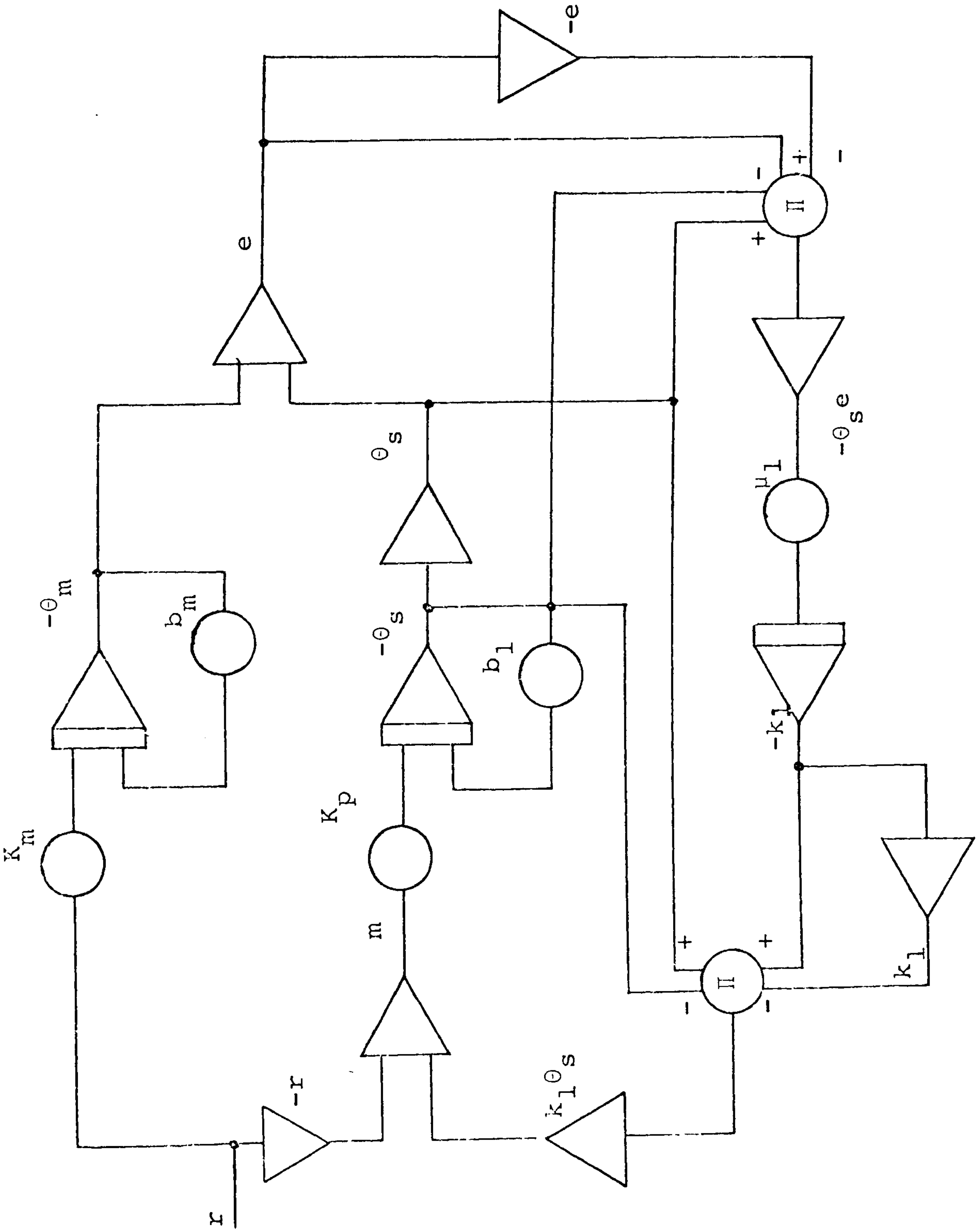
2. <u>Analog Simulation Drawing of First Order System With sat Function Control Law</u>



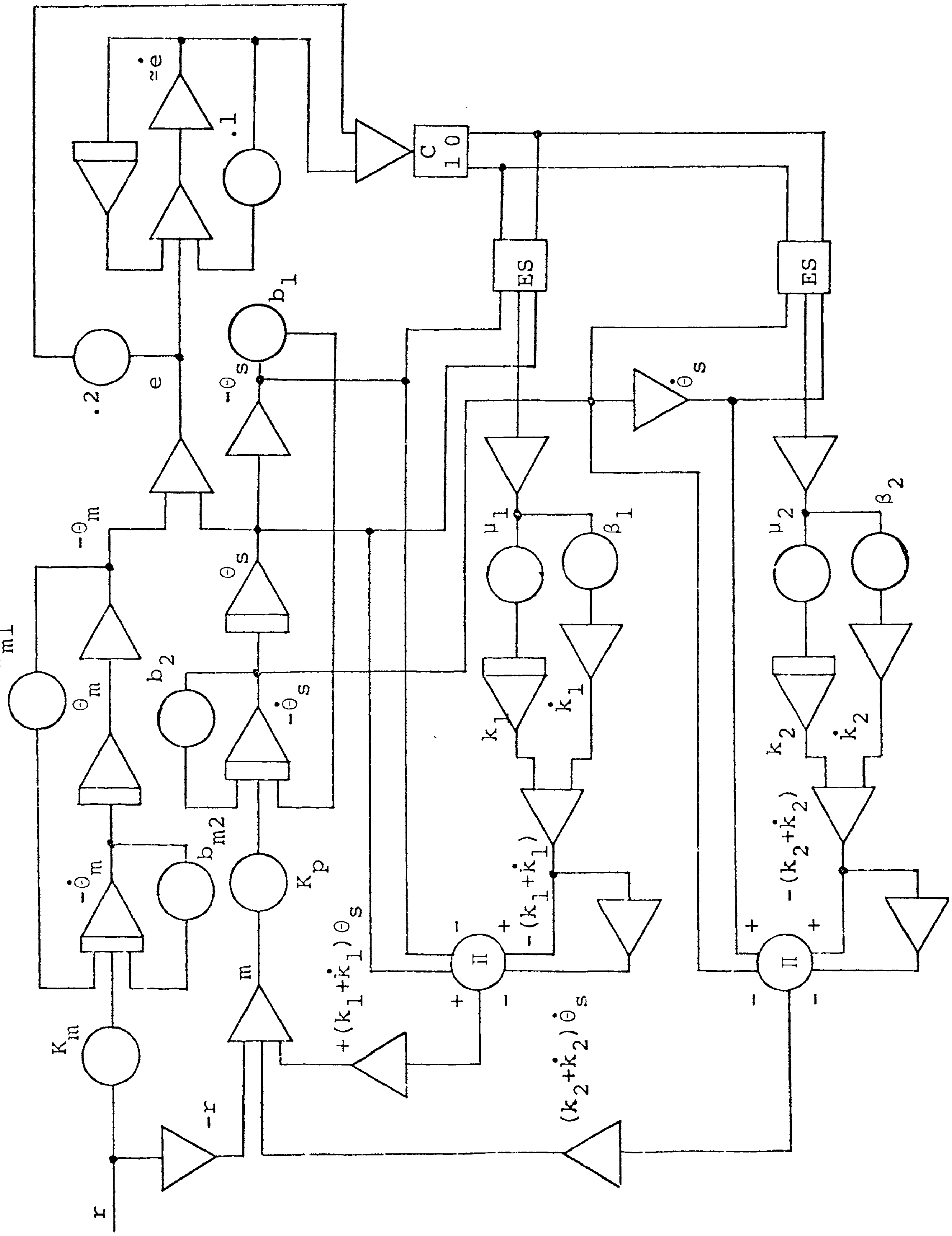
3. Analog Simulation Drawing of First Order System With sgn Function Control Law



4. Analog Simulation Drawing of First Order System With Product Control Law



5. Analog Simulation Drawing of Second Order System With SGN Control Law



VITA

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